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Indeterministic Objects in the Category of Effect Algebras and the Passage to the Semiclassical Limit

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We consider a category of effect algebras and formulate an abstract-hidden variables problem for an object of this category. A notion of indeterministic object is introduced as of an object which induces a Kochen–Specker-type contradiction. A sufficient condition for an object to be indeterministic is derived. An abstract algebraic point of view on a no-hidden variables example constructed by Mermin is given. The notion of a passage to the semiclassical limit is analyzed and refined.

KEY WORDS: effect algebra; Kochen–Specker; semiclassical limit.

1. INTRODUCTION

Quantum theory can be viewed as a mathematical formalism providing a framework for the construction of probabilistic models for certain types of physical experiments. The "unusual" or the "contradicting common sense" properties of quantum theory arise mainly from the fact that it admits incompatible measurements. The aim of the current article is to analyze some general implications of the incompatibility of measurements for the mathematical formalism of an abstract physical theory.

From the very beginning, it is necessary to point out, that we are not going to discuss a metaphysical problem of constructing some "objective reality" that stands behind an experimental data. That is why we avoid such notions like *properties* and *states* of a physical system (and by that the notion of a physical *system* itself), and deal with the problem from the position of mathematical pragmatism. A measuring device $\mathbf{A}$ is viewed as a "black box" with an indicator and there is a *finite* set $I_{\mathbf{A}}$ of all possible indications it can show. Naively, a description of a physical experiment consists in the following. There is a set $\mathcal{R}$ of *recipes* of preparation for an experiment and a set $\mathcal{D}$ of measuring *devices*. The set $\mathcal{D}$ is equipped with a

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relation $\mathcal{K} \subset \mathcal{D} \times \mathcal{D}$ – the compatibility relation between the measuring devices. Denote the set of all *finite* subsets of $\mathcal{D}$ consisting of pairwise compatible devices as $\mathcal{P}_{\mathcal{K}}^{f}(\mathcal{D})$. A description of what has been done in a given experiment is a pair $(U, R) \in \mathcal{P}_{\mathcal{K}}^{f}(\mathcal{D}) \times \mathcal{R}$. A description of what is the outcome of an experiment with (U, R) is an element $\phi \in \mathcal{O}_U$ of a direct product $\mathcal{O}_U := \Pi_{\mathbf{A}\in U} I_{\mathbf{A}}$. Consider a disjoint union $\mathcal{T} := \sqcup_{U\in\mathcal{P}_{\mathcal{K}}^{f}(\mathcal{D})}\mathcal{O}_U$ and for every $U \in \mathcal{P}_{\mathcal{K}}^{f}(\mathcal{D})$ denote by $i_U : \mathcal{O}_U \rightarrowtail \mathcal{T}$ the natural injection.

Imagine that one has a sample series of experiments $\mathcal{Z}$. For every $(U, R) \in \mathcal{P}_{\mathcal{K}}^{f}(\mathcal{D}) \times \mathcal{R}$ denote by $\mathcal{Z}_{U,R} \subset \mathcal{Z}$ a subset of all the experiments which have a recipe of preparation R and involve a set of measuring devices U. For every $\phi \in \mathcal{O}_U$ extract from this set a subset $\mathcal{Z}_{U,R}^{\phi} \subset \mathcal{Z}_{U,R}$ of all the experiments with an outcome ϕ. If $\#\mathcal{Z}_{U,R} \neq 0$ one may consider a quantity $\#\mathcal{Z}_{U,R}^{\phi}/\#\mathcal{Z}_{U,R}$ – the relative frequency of an outcome ϕ corresponding to (U, R). One assumes that for every (U, R) this quantity has a limit as the number $\#\mathcal{Z}_{U,R} \to \infty$. We shall call this limit the *expected relative frequency*. It means that there exists a function

$$F(\cdot|\cdot) : \mathcal{T} \times \mathcal{R} \to [0, 1], \qquad (T, R) \mapsto F(T|R),$$

such that $F(i_U(\phi)|R)$ has a meaning of the expected relative frequency of an outcome $\phi \in \mathcal{O}_U$ corresponding to $(U, R) \in \mathcal{P}_{\mathcal{K}}^{f}(\mathcal{D}) \times \mathcal{R}$.

One can naturally define equivalence relations on the sets $\mathcal{T}$ and $\mathcal{R}$: $R_1, R_2 \in \mathcal{R}$ are declared to be equivalent iff $\forall T \in \mathcal{T}$ one has $F(T|R_1) = F(T|R_2)$; $T_1, T_2 \in \mathcal{T}$ are declared to be equivalent iff $\forall R \in \mathcal{R}$ one has $F(T_1|R) = F(T_2|R)$. Denote the factorizations of $\mathcal{T}$ and $\mathcal{R}$ with respect to these equivalence relations as $\mathfrak{D}$ and $\mathfrak{R}$ respectively. The function F induces a function

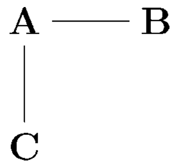
$$\Phi(\cdot|\cdot) : \mathfrak{D} \times \mathfrak{R} \to [0, 1], \qquad (P, S) \mapsto \Phi(P \mid S).$$

The triple $(\mathfrak{D}, \mathfrak{R}, \Phi)$ should play a key role in all the theory.

The Hilbert space formalism of quantum mechanics allows an element $P \in \mathfrak{D}$ to be identified with an orthogonal projector $\widehat{\pi}_P$ onto some closed linear manifold in the Hilbert space $\mathcal{H}$ associated to the physical system. Every element $S \in \mathfrak{R}$ is then identified with a self-adjoint non-negative operator $\widehat{\rho}_S$ with a unit trace, and the value $\Phi(P|S)$ is given by the formula $\Phi(P|S) = Tr(\widehat{\rho}_S\widehat{\pi}_P)$. One may try to think that two equivalent $T_1, T_2 \in \mathcal{T}$ correspond to "looking at the same property" of a physical system, and that two equivalent $R_1, R_2 \in \mathcal{R}$ correspond to "creating the same state" in a physical system. As was already mentioned, we avoid the notions of states and properties. The reason is that we focus the attention on the notion of incompatibility of measuring devices without giving a cause for the assertions like "a physical system does not have a property before it is measured," "a physical system can exist in two states at one time," etc., which seem absurd.

Consider an example illustrating incompatibility. Let **A**, **B**, and **C** denote three measuring devices with sets of possible indications $I_{\mathbf{A}}$, $I_{\mathbf{B}}$, and $I_{\mathbf{C}}$ respectively.

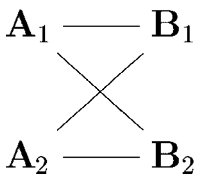
Assume, that **A** is compatible with **B** and with **C**, but **B** is not compatible with **C**. This is illustrated by the diagram:



In this case it is impossible to organize an experiment with a simultaneous measurement of **A**, **B**, and **C**. Nevertheless, one is tempted to speculate about the following. It seems natural to think, that every experiment splits into two stages: (1) preparation process and (2) measurement process. In this case the experimentator may decide what to measure *after* performing a preparation. Suppose, for example, that in some experiment one has measured $\{\mathbf{A}, \mathbf{B}\}$ and has obtained an outcome $(a, b) \in I_{\mathbf{A}} \times I_{\mathbf{B}}$. Imagine, that the experimentator could go back in time to the moment between the two stages of this experiment and change his decision about what to measure from $\{\mathbf{A}, \mathbf{B}\}$ to $\{\mathbf{A}, \mathbf{C}\}$. Then he would obtain some indications a' and c' for the devices **A** and **C** respectively. It seems natural to think, that he must get $a' = a$. This leads to an idea, that a result of every preparation process should be described by $(a, b, c) \in I_{\mathbf{A}} \times I_{\mathbf{B}} \times I_{\mathbf{C}}$. On the other hand, without the "traveling in time" one can extract only a part of (a, b, c) via a measurement process. This leads to the question: given a series of experiments $\mathcal{Z}$, is it possible to derive some knowledge about the "probability" to have (a, b, c) as a result of a preparation $R \in \mathcal{R}$ from the knowledge of all possible relative frequencies? Of course, it is necessary to clarify mathematically what the word "probability" means here. For example, is it possible to write some inequalities for this quantity, based on the knowledge of all possible binary relative frequencies? As will be shown in subsequent sections, the analysis of this sort of questions reduces to an abstract-hidden variables problem. The term "hidden variables" was introduced by D. Bohm in (Bohm, 1952). J. von Neumann analyzed this problem in his book (von Neumann, 1955). Better versions of "no-hidden variables" theorems were obtained by Bell (1966) and Kochen and Specker (1967). A modern point of view on Kochen–Specker theorem based on topos theory can be found in Isham and Butterfield (1998), Butterfield and Isham (1999, 2002),and Hamilton *et al.* (2000). For a detailed analysis of the foundations of quantum theory one can refer, for example, to Aerts *et al.* (1999) and Omnès (1988a,b,c, 1989).

2. CONTRADICTION WITH COMMON SENSE

In this section we would like to give an example that shows the importance of the notion of incompatibility. We continue to use the notations from the introduction. Suppose that one considers a subset of $\mathcal{D}$ consisting of four measuring devices $\mathbf{A}_1, \mathbf{B}_1, \mathbf{A}_2, \mathbf{B}_2 \in \mathcal{D}$ such that each of the devices has only two possible indications: 0 and 1. Assume, that the relations of compatibility are described by the diagram:



(1)

It implies that working with these devices one may only consider experiments, which involve a single measuring device $\mathbf{A}_i$ or $\mathbf{B}_j$, or a pair of devices of the form $\{\mathbf{A}_i, \mathbf{B}_j\}$ $(i, j = 1, 2)$. The listed variants correspond to all the completely connected subcomponents of the graph. Let $\mathbf{A}_i^{(\mu)} \in \mathfrak{D}$ denote an image of $(\mu) \in \mathcal{O}_{\{\mathbf{A}_i\}}$ under the composition of natural maps $\mathcal{O}_{\{\mathbf{A}_i\}} \rightarrowtail \mathcal{T}$ and $\mathcal{T} \twoheadrightarrow \mathfrak{D}$. Introduce similar notations $\mathbf{B}_j^{(\nu)}$, $(\mathbf{AB})_{i,j}^{(\mu,\nu)} \in \mathfrak{D}$ for the images of $(\nu) \in \mathcal{O}_{\{\mathbf{B}_j\}}$ and $(\mu, \nu) \in I_{\mathbf{A}_i} \times I_{\mathbf{B}_j} = \mathcal{O}_{\{\mathbf{A}_i, \mathbf{B}_j\}}$ respectively $(i, j = 1, 2;\ \mu, \nu = 0, 1)$.

The function $\Phi : \mathfrak{D} \times \mathfrak{R} \to [0, 1]$ defining the expected relative frequencies is not arbitrary. In particular, it has to satisfy for every $S \in \mathfrak{R}$ the following conditions:

$$\sum_{\mu'=0,1} \Phi\big(\mathbf{A}_i^{(\mu')}|S\big) = 1, \qquad \sum_{\nu'=0,1} \Phi\big(\mathbf{B}_j^{(\nu')}|S\big) = 1,$$

and

$$\Phi\big(\mathbf{A}_i^{(\mu)}|S\big) = \textstyle\sum_{\nu'=0,1} \Phi\big((\mathbf{AB})_{i,j}^{(\mu,\nu')}|S\big),$$
$$\Phi\big(\mathbf{B}_j^{(\nu)}|S\big) = \textstyle\sum_{\mu'=0,1} \Phi\big((\mathbf{AB})_{i,j}^{(\mu',\nu)}|S\big),$$

where $i, j = 1, 2;\ \mu, \nu = 0, 1$. Note, that there are similar formulas (the *normalization* and *consistency* conditions) for every $U \in \mathcal{P}_{\mathcal{K}}^{f}(\mathcal{D})$.

For every $S \in \mathfrak{R}$ we have a function $\Phi(\cdot|S) : \mathfrak{D} \to [0, 1]$. One can put the following general question: what is possible to say about the set of all functions of the form $\Phi(\cdot|S)$ obtained as S runs over the entire $\mathfrak{R}$? It may seem natural, that for every $S \in \mathfrak{R}$ a function $\Phi(\cdot|S)$ can be treated as a probability measure. We give an example that shows that this doesn't have to be the case.

Assume that there exists $S \in \mathfrak{R}$ and the four devices mentioned above such that one has: $\Phi(\mathbf{A}_1^{(1)}|S) = \Phi(\mathbf{A}_2^{(1)}|S) = 1/2$, $\Phi(\mathbf{B}_1^{(1)}|S) = \Phi(\mathbf{B}_2^{(1)}|S) = 1/2$ and $\Phi((\mathbf{AB})_{1,1}^{(1,1)}|S) = 1/8$, $\Phi((\mathbf{AB})_{1,2}^{(1,1)}|S) = \Phi((\mathbf{AB}_{2,1}^{(1,1)}|S) = 3/8$, $\Phi((\mathbf{AB})_{2,2}^{(1,1)}|S) = 1/2$. It will be shown below that in quantum mechanics this assumption can be fulfilled. The rest of the values can be found from the normalization and consistency conditions. This gives $\Phi(\mathbf{A}_i^{(\mu)}|S) = \Phi(\mathbf{B}_j^{(\nu)}|S) = 1/2$ and a Table I for $\Phi((\mathbf{AB})_{i,j}^{(\mu,\nu)}|S)$ $(i, j = 1, 2$ and $\mu, \nu = 0, 1)$. Note, that all the calculated quantities fall into $[0, 1]$.

Table I. A Cell at the Intersection of the Row ij and the Column $\mu\nu$ Contains the Expected Relative Frequency $\Phi((\mathbf{AB})_{i,j}^{(\mu,\nu)}|S)$

	00	01	10	11
11	1/8	3/8	3/8	1/8
12	3/8	1/8	1/8	3/8
21	3/8	1/8	1/8	3/8
22	1/2	0	0	1/2

Note, that the last line of the table creates an illusion of an existence of a conservation law.

One is tempted to interpret these expected relative frequencies as probabilities. It means, that one tries to think, that there exists a probability space $(\Omega_S, \mathcal{F}_S, P_S)$ and events $A_1^S, A_2^S, B_1^S, B_2^S \in \mathcal{F}_S$, such that $P_S(A_i^S) = \Phi(\mathbf{A}_i^{(1)}|S)$, $P_S(B_j^S) = \Phi(\mathbf{B}_j^{(1)}|S)$ and $P_S(A_i^S B_j^S) = \Phi((\mathbf{AB})_{i,j}^{(1,1)}|S)$, where $i, j = 1, 2$. The expected relative frequency $\Phi(\mathbf{A}_i^{(0)}|S)$ will correspond to $P_S(\overline{A_i^S})$ and similarly $\Phi(\mathbf{B}_j^{(0)}|S)$ will correspond to $P_S(\overline{B_j^S})$, $i, j = 1, 2$. As for the expected relative frequencies with two measuring devices, one will have, for example, $\Phi((\mathbf{AB})_{1,2}^{(1,0)}|S) = P_S(A_1^S\overline{B_2^S})$. The other entries of the table are converted into probabilities by analogy.

Suppose the mentioned interpretation is right. Let us omit in what follows the index S in notations $A_1^S, A_2^S, B_1^S, B_2^S$ and P_S. Then the normalization and consistency conditions may be perceived as the implications of standard formulas of probability theory: $P(x) + P(\bar{x}) = 1$ and $P(xy) + P(x\bar{y}) = P(x)$. Nevertheless, such an interpretation contains a contradiction, which appears when one tries to derive some implications about the events, which cannot be seen in the experiment. Namely, consider the probabilities $P(A_1B_1\overline{A}_2\overline{B}_2)$ and $P(\overline{A}_1\overline{B}_1A_2B_2)$. Applying the two formulas from probability theory mentioned above and taking into account that $P(x) \geq 0$, one obtains the following inequalities:

$$
\begin{aligned}
P(A_1B_1\overline{A}_2\overline{B}_2) &= P(A_1B_1) - P(A_1B_1\overline{A}_2B_2) - P(A_1B_1A_2\overline{B}_2) \\
&\quad - P(A_1B_1A_2B_2) \geq 0, \\
P(\overline{A}_1\overline{B}_1A_2B_2) &= P(A_2B_2) - P(\overline{A}_1B_1A_2B_2) - P(A_1\overline{B}_1A_2B_2) - P(A_1B_1A_2B_2) \\
&= P(A_2B_2) - P(B_1A_2B_2) - P(A_1A_2B_2) + P(A_1B_1A_2B_2) \geq 0.
\end{aligned}
$$

Note, that $P(A_1B_1A_2B_2)$ enters these two inequalities with different signs. Expressing this term, one gets

$$
\begin{aligned}
P(A_1B_1A_2B_2) &\leq P(A_1B_1) - P(A_1B_1\overline{A}_2B_2) - P(A_1B_1A_2\overline{B}_2), \\
P(A_1B_1A_2B_2) &\geq -P(A_2B_2) + P(B_1A_2B_2) + P(B_1A_2B_2).
\end{aligned}
$$

Denote the right-hand side of the first inequality as α and the right-hand side of the second inequality as β. The contradiction between the two inequalities arises iff $\alpha < \beta$ (note, that a strict inequality is required). β can be written as

$$\beta = -P(A_2B_2) + \left[P(B_1A_2) - P(B_1A_2\overline{B}_2)\right] + \left[P(A_1B_2) - P(A_1\overline{A}_2B_2)\right].$$

Combining the term $P(A_1B_1A_2\overline{B}_2)$, coming from α, with the term $P(B_1A_2\overline{B}_2)$, coming from β, and the term $P(A_1B_1\overline{A}_2B_2)$, coming from α, with the term $P(A_1\overline{A}_2B_2)$, coming from β, one reduces $\alpha < \beta$ to the form

$$P(A_1\overline{B}_1\overline{A}_2B_2) + P(\overline{A}_1B_1A_2\overline{B}_2) < -P(A_1B_1) - P(A_2B_2) + P(B_1A_2) + P(A_1B_2).$$

Now, taking into account the inequality from the standard probability theory $P(xy) \leq P(x)$, one obtains a *sufficient* condition for the contradiction between the requirements $P(A_1B_1\overline{A}_2\overline{B}_2) \geq 0$ and $P(\overline{A}_1\overline{B}_1A_2B_2) \geq 0$:

$$\begin{aligned} &\min\left\{P(A_1\overline{B}_1), P(\overline{A}_2B_2), P(\overline{B}_1\overline{A}_2), P(A_1B_2)\right\} \\ &+\min\left\{P(\overline{A}_1B_1), P(A_2\overline{B}_2), P(B_1A_2), P(\overline{A}_1\overline{B}_2)\right\} \\ &< -P(A_1B_1) - P(A_2B_2) + P(B_1A_2) + P(A_1B_2). \end{aligned} \tag{2}$$

It turns out, that for the data described above, the inequality (2) really takes place: $0 < 1/8$, giving rise to the mentioned contradiction.

The obtained contradiction implies that either the mentioned $\Phi(\cdot|S)$ cannot be treated as a probability measure, or the initial assumption about the existence of $S \in \mathfrak{R}$ with the required $\Phi(\cdot|S)$ cannot be valid. Invoking the speculation at the end of the introduction, one is inclined to accept the latter variant. Nevertheless the surprising fact about quantum mechanics is that its mathematical formalism admits a realization of $\Phi((\mathbf{AB})_{i,j}^{(\mu,\nu)}|S)$ $(i, j = 1, 2, \mu, \nu = 0, 1)$. It means that one has to accept the first possibility. To persuade oneself in this, one can do the following. Take any orthonormal basis $\{\varphi_0, \varphi_1\}$ in $\mathbb{C}^2$. Denote $\psi_0 = (\varphi_0 - \sqrt{3}\varphi_1)/2$, $\psi_1 = (\sqrt{3}\varphi_0 + \varphi_1)/2$, $\chi_0 = (\sqrt{3}\varphi_0 - \varphi_1)/2$, $\chi_1 = (\varphi_0 + \sqrt{3}\varphi_1)/2$. Let $\Psi = (\varphi_0 \otimes \varphi_0^* + \varphi_1 \otimes \varphi_1^*)/\sqrt{2} \in \mathbb{C}^2 \otimes \mathbb{C}^2$ (the * denotes the complex conjugation). Note, that Ψ does not depend on the choice of $\{\varphi_0, \varphi_1\}$. Let $\widehat{\rho}$ be an orthogonal projector onto $\mathbb{C}\Psi$. Let $\widehat{a}_1^{(\mu)} = \widehat{\pi}_{\varphi_\mu} \otimes id$, $\widehat{a}_2^{(\mu)} = \widehat{\pi}_{\psi_\mu} \otimes id$, $\widehat{b}_1^{(\nu)} = id \otimes \widehat{\pi}_{\chi_\nu}$ and $\widehat{b}_2^{(\nu)} = id \otimes \widehat{\pi}_{\psi_\nu^*}$ ($\widehat{\pi}_\varphi$ denotes an orthogonal projector onto $\mathbb{C}\varphi$, $\varphi \in \mathbb{C}^2$). Using the formula for the inner product $(u \otimes v, \Psi) = (u, v^*)/\sqrt{2}$ (where $u, v \in \mathbb{C}^2$), one verifies that $Tr(\widehat{\rho}\,\widehat{a}_i^{(\mu)}) = Tr(\widehat{\rho}\,\widehat{b}_j^{(\nu)}) = 1/2$ and that $Tr(\widehat{\rho}\widehat{a}_i^{(\mu)}\widehat{b}_j^{(\nu)})$ is equal to the $((i, j), (\mu, \nu))$-th entry of the Table I.

3. HIDDEN VARIABLES PROBLEM

One may argue, that the contradiction described in the previous section, is determined by the fact, that one tries to speculate about the events, such as $A_1 B_1 \overline{A}_2 \overline{B}_2$ and $\overline{A}_1 \overline{B}_1 A_2 B_2$, which cannot be observed in principle due to the restrictions imposed by the incompatibility of measuring devices (rf. (4)). One is tempted to look at this fact in the following way. There exists some measurable space $(\Omega, \mathcal{F})$ equipped with some σ-additive map $P(\cdot) : \mathcal{F} \to \mathbb{C}$ (a $\mathbb{C}$-valued measure) satisfying $P(\Omega) = 1$. Note, that $P(\cdot)$ is not required to be a map to $[0, 1]$. The set of measuring devices $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{B}_1, \mathbf{B}_2\}$ is *injectively* mapped into $\mathcal{F}$ in such a way, that for every nontrivial event that one is able to observe in experiment (i.e. the events $A_i, \overline{A}_i, B_j, \overline{B}_j, A_i B_j, A_i \overline{B}_j, \overline{A}_i B_j$, and $\overline{A}_i \overline{B}_j, i, j = 1, 2$) one gets a measure, which belongs to $[0, 1]$. For other, "virtual" events, the measure can be any number. (Note that the example considered in the previous section can really be interpreted this way: take $\Omega = \{0, 1\}^4, \mathcal{F} = \mathcal{P}(\Omega), A_i = \{(\alpha_1, \alpha_2, \beta_1, \beta_2) \in \Omega | \alpha_i = 1\}$, $B_j = \{(\alpha_1, \alpha_2, \beta_1, \beta_2) \in \Omega | \beta_j = 1\}, i, j = 1, 2$; the assigning of values to $P(\Omega)$, $P(A_i)$, $P(B_j)$, and $P(A_i B_j)$ gives a system of 9 linear equations with respect to the measures of 16 points $(\alpha_1, \alpha_2, \beta_1, \beta_2) \in \Omega$, which happens to have many solutions). On the other hand, in general, the situation turns out to be more complicated: it is possible to construct examples of experimental data for which the measurable space does not exist at all.

In this section we shall formulate an abstract-hidden variables problem in terms of homological algebra. Let D be a set equipped with some relation E and a map $\cdot \oplus \cdot : E \to D, (x, y) \mapsto x \oplus y$. Assume that the following conditions hold:

1) $\forall (x, y) \in E : x \oplus y = y \oplus x$.
2) $\forall (x, y) \in E \, \forall z, (x \oplus y, z) \in E : (x \oplus y) \oplus z = x \oplus (y \oplus z)$.
3) $\forall x, y, z \in D, (x, y) \in E, (x, z) \in E : x \oplus y = x \oplus z \Rightarrow y = z$.
4) $\exists \mathbf{0} \in D \, \forall x \in D : x = x \oplus \mathbf{0}$.
5) $\exists \mathbf{1} \in D \, \forall x \in D \exists x^* \in D : \mathbf{1} = x \oplus x^*$.
6) $\forall (x, y) \in E : x \oplus y = \mathbf{0} \Rightarrow x = \mathbf{0} \,\&\, y = \mathbf{0}$.

Note, that the axioms imply, in particular, that D is not empty and that E is a symmetric relation. Note, that if $E = D \times D$, then $(D; \oplus)$ is a commutative monoid. One can show, that $\mathbf{0}, \mathbf{1}$, and x^* (for every x) are defined in a unique way and that $\mathbf{0}^* = \mathbf{1}$ (the latter follows from $\mathbf{0} \oplus \mathbf{1} = \mathbf{1}$). We shall refer to the data $(D, E, \oplus)$ satisfying these six axioms as to *effect algebra*. Note, that the term "effect algebra" was suggested in Foulis and Bennett (1994). We use a slightly different, but an equivalent set of axioms to the one formulated in the mentioned article. Note, that the notion of effect algebra is close to the notion of *orthoalgebra*, which was investigated in Foulis *et al.* (1992). It is convenient to introduce a category of effect algebras $\mathcal{C}$. Given an object X, one denotes the corresponding $\mathbf{0}$ and $\mathbf{1}$ elements as $\mathbf{0}_X$ and $\mathbf{1}_X$ and refers to them as to *zero* and *unit* elements for X respectively. The set of morphisms $\mathrm{Hom}_{\mathcal{C}}(X_1, X_2)$ from an object $X_1 = (D_1, E_1, \oplus_1)$ to an object $X_2 = (D_2, E_2, \oplus_2)$ is defined as follows. A morphism $f : X_1 \to X_2$ is given by the data consisting just of a map $m : D_1 \to D_2$, which satisfies the conditions:

α) $\forall (x, y) \in E_1 : m(x \oplus y) = m(x) \oplus m(y)$, i.e. $m : D_1 \to D_2$ induces a map $\widetilde{m} : E_1 \to E_2$ and the diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{\oplus_1} & D_1 \\ {\scriptstyle \widetilde{m}}\downarrow & & \downarrow{\scriptstyle m} \\ E_2 & \xrightarrow[\oplus_2]{} & D_2 \end{array}$$

commutes.

β) $m(\mathbf{1}_{X_1}) = \mathbf{1}_{X_2}$.

The composition of two morphisms is given by the composition of the corresponding maps. Note, that one does not require $m(\mathbf{0}_{X_1}) = \mathbf{0}_{X_2}$, since this fact follows from other axioms. The requirement $m(\mathbf{1}_{X_1}) = \mathbf{1}_{X_2}$ can be viewed as a normalization condition. Note, that there exists a natural forgetful functor $for : \mathcal{C} \to$ **Set** to the category of sets. Let us also mention, that the category $\mathcal{C}$ admits direct products (this fact is to play its role in the context of the consistent histories approach to quantum theory initiated in Griffits (1984); Omnès (1988); Gell-Mann and Hartle (1992). Note, that the notion of orthoalgebra is used in the formulation of the general formalism of consistent histories in Isham (1997) and Isham and Linden (1994). If there is a family of objects, $\{X_t\}_{t\in T}$, $X_t = (D_t, E_t, \oplus_t)$, T is some index set, then there exists a direct product $X = \Pi_{t\in T} X_t$ given by the data $X = (D, E, \oplus)$, $D := \Pi_{t\in T} D_t$, $E := \{(\{x_t\}_t, \{y_t\}_t) | \forall t \in T : (x_t, y_t) \in E_t\}$, $\{x_t\}_t \oplus \{y_t\}_t := \{x_t \oplus_t y_t\}_t$. For this X one will have $\mathbf{0}_X = \{\mathbf{0}_{X_t}\}_t$, $\mathbf{1}_X = \{\mathbf{1}_{X_t}\}_t$, $(\{x_t\}_t)^* = \{x_t^*\}_t$.

Consider arbitrary $X \in \mathrm{Ob}(\mathcal{C})$ ($\mathrm{Ob}(\mathcal{C})$ denotes the class of objects in $\mathcal{C}$), $X = (D, E, \oplus)$. One associates to $\oplus$ a relation $\preccurlyeq$ on D:

$$x \preccurlyeq y \rightleftharpoons \exists x_1 : x \oplus x_1 = y.$$

Proposition 3.1. *For every object $X = (D, E, \oplus)$ of $\mathcal{C}$ the following is true:*

1) *The relation $\preccurlyeq$ is a partial order on D.*
2) *The elements $\mathbf{0}_X$ and $\mathbf{1}_X$ play the roles of the minimal and maximal elements of $(D; \preccurlyeq)$ respectively.*
3) *The operation $x \mapsto x^*$ is an involution on $(D; \preccurlyeq)$, i.e. $\forall x \in D : x^{**} = x$ and $\forall x, y \in D : x \preccurlyeq y \Leftrightarrow y^* \preccurlyeq x^*$.*
4) *The relation E can be represented as $E = \{(x, y) | x \preccurlyeq y^*\}$.*

Proof: The proof of these facts is easy and follows directly from the axioms. We give it only for the convenience of the reader.

1) Indeed, $x \preccurlyeq x$, since $x \oplus \mathbf{0}_X = x$. Thus $\preccurlyeq$ is reflexive. If $x \preccurlyeq y$ and $y \preccurlyeq z$, then $x \oplus x_1 = y$ and $y \oplus y_1 = z$ for some y_1, z_1. Using associativity of $\oplus$, one gets $z = (x \oplus x_1) \oplus y_1 = x \oplus (x_1 \oplus y_1)$. It follows, that $x \preccurlyeq z$. Thus $\preccurlyeq$ is transitive. Let us show now that $\preccurlyeq$ is antisymmetric. Assume $x \preccurlyeq y \& y \preccurlyeq x$. It means, that $x \oplus x_1 = y$ and $y \oplus y_1 = x$ for some x_1, y_1. It follows, that $x \oplus (x_1 \oplus y_1) = x$, and this gives $x_1 \oplus y_1 = \mathbf{0}_X$. One concludes, that $x_1 = y_1 = \mathbf{0}_X$. It follows, that $x = y$, and thus, $\preccurlyeq$ is antisymmetric.
2) We have shown, that $\preccurlyeq$ is a partial order on D. $\mathbf{0}_X$ is a minimal element, since for any x, $\mathbf{0}_X \oplus x = x$, i.e. $\mathbf{0}_X \preccurlyeq x$. $\mathbf{1}_X$ is a maximal element, since for all x, one has $x \oplus x^* = \mathbf{1}_X$, what implies, that $x \preccurlyeq \mathbf{1}_X$.
3) Take any $x \in D$. We have: $x^* \oplus x^{**} = \mathbf{1}_X = x \oplus x^* \Rightarrow x^{**} = x$. Take any $x, y \in D$ and suppose $x \preccurlyeq y$. It means, that $x \oplus x_1 = y$ for some x_1. One has: $(x \oplus x_1) \oplus y^* = y \oplus y^* = \mathbf{1}_X$. Using associativity of $\oplus$, one gets $x \oplus (x_1 \oplus y^*) = \mathbf{1}_X$, i.e. $x^* = x_1 \oplus y^*$. It means, that $y^* \preccurlyeq x^*$.
4) Denote $E' := \{(x, y) | x \preccurlyeq y^*\}$. Let us show that $E \subset E'$. Take any $(x, y) \in E$. One has $(x \oplus y) \oplus (x \oplus y)^* = \mathbf{1}_X$. Using associativity and commutativity of $\oplus$, one gets $(x \oplus (x \oplus y)^*) \oplus y = \mathbf{1}_X$. On the other hand $\mathbf{1}_X = y^* \oplus y$. It means, that $x \oplus (x \oplus y)^* = y^*$. It follows, that $x \preccurlyeq y^*$, and thus $E \subset E'$. Let us show now, that $E' \subset E$. Take any (x, y), such that $x \preccurlyeq y^*$. Then $\exists x_1 : x \oplus x_1 = y^*$. Since $y \oplus y^*$ is well defined for any y, one has $(x \oplus x_1, y) \in E$. According to the axioms of associativity and commutativity of $\oplus$, $(x \oplus x_1) \oplus y = x_1 \oplus (x \oplus y)$. In particular, this implies, that $(x, y) \in E$. □

Note, that the last axiom was necessary in the proof of antisymmetry of the relation $\preccurlyeq$. One calls $\preccurlyeq$ a *standard* partial order.

Proposition 3.2. *Let $f : X \to Y$ be a morphism in $\mathcal{C}$, given by a map $m = for(f)$. It is claimed, that m respects the standard partial order and the involution, i.e., $\forall x \in for(X) : m(x^*) = m(x)^*$, and $\forall x, y \in for(X) : x \preccurlyeq y \Rightarrow m(x) \preccurlyeq m(y)$.*

Proof: Take any x. One has: $\mathbf{1}_Y = m(\mathbf{1}_X) = m(x \oplus x^*) = m(x) \oplus m(x^*)$. $\Rightarrow m(x^*) = m(x)^*$.

Suppose $x \preccurlyeq y$. It means, that $x \oplus x_1 = y$ for some x_1. One has: $m(y) = m(x \oplus x_1) = m(x) \oplus m(x_1)$. This implies, that $m(x) \preccurlyeq m(y)$. □

Let us give some examples of objects of the category $\mathcal{C}$, which will play a role in further sections.

Example 3.1. Let $\mathcal{H}$ be a Hilbert space. Denote by $L(\mathcal{H})$ the set of all closed linear systems in $\mathcal{H}$. Let $E = \{(P, Q) \in L(\mathcal{H}) \times L(\mathcal{H}) | P \perp Q\}$ and define $\oplus$ as the closure of the linear span. The data $(L(\mathcal{H}), E, \oplus)$ defines and object in $\mathcal{C}$, which will be denoted as $\mathbb{L}(\mathcal{H})$.

Example 3.2. Let $(\Omega, \mathcal{F})$ be a measurable space. Let E be a relation on the σ-algebra $\mathcal{F}$, $E = \{(P, Q) | P \cap Q = \emptyset\}$. The data $(\mathcal{F}, E, \sqcup)$ defines an object of $\mathcal{C}$ which will be denoted as $\mathbb{W}(\Omega, \mathcal{F})$.

Example 3.3. Denote by E a relation on $[0, 1]$, $E = \{(\alpha, \beta) | \alpha + \beta \leq 1\}$. The data $([0, 1], E, +)$ defines an object of $\mathcal{C}$ which will be denoted as $\mathbb{I}$. Note, that every probability measure on $(\Omega, \mathcal{F})$ defines a morphism $\mathbb{W}(\Omega, \mathcal{F}) \to \mathbb{I}$.

Let $X \in \mathrm{Ob}(\mathcal{C})$, $X = (D, E, \oplus)$. It turns out that the existence of an involution * on $(D; \preccurlyeq)$ allows to extend the quantum mechanical notion of compatibility to any X.

Definition 3.4. Two elments $x, y \in D$ are called *compatible* in X, if the following conditions hold

1) $\exists \inf\{x, y\} =: x \wedge y \& \exists \sup\{x, y\} =: x \vee y$.
2) The following formulas are valid:

$$(x \wedge y) \oplus x^* = y \oplus (x \vee y)^*,$$

$$(x \wedge y) \oplus y^* = x \oplus (x \vee y)^*.$$

Note, that $(x \vee y)^* = x^* \wedge y^*$.

Proposition 3.3. *Let $\mathcal{H}$ be a Hilbert space. Two elments $P, Q \in L(\mathcal{H})$ are compatible in $\mathbb{L}(\mathcal{H})$ iff the orthogonal projectors $\widehat{\pi}_P$ and $\widehat{\pi}_Q$ on P and Q respectively commute.*

Proof: Let us first prove one general auxiliary fact. Let $X = (D, E, \oplus) \in \mathrm{Ob}(\mathcal{C})$ and let $x, y \in D$ be compatible. Define ξ and η from the formulae: $x = \xi \oplus (x \wedge y)$, $y = \eta \oplus (x \wedge y)$. One claims, that $\xi \preccurlyeq \eta^*$. Indeed, let us define ξ_1 from $y^* = \inf\{x^*, y^*\} \oplus \xi_1$. It follows, that

$$\inf\{x, y\} \oplus y^* = \inf\{x, y\} \oplus \xi_1 \oplus \inf\{x^*, y^*\},$$

$$x \oplus \inf\{x^*, y^*\} = \inf\{x, y\} \oplus \xi \oplus \inf\{x^*, y^*\}.$$

Taking into account the definition of compatible elements and the property $\sup\{x, y\}^* = \inf\{x^*, y^*\}$ (which is a general property for a partially ordered set equipped with involution), one concludes, that $\xi = \xi_1$. Similarly, if one defines

η_1 from $x^* = \inf\{x^*, y^*\} \oplus \eta_1$, one deduces, that $\eta_1 = \eta$. Now, one has $\xi \preccurlyeq x$ and $\eta \preccurlyeq x^*$. It follows, that $\xi \preccurlyeq x \preccurlyeq \eta^*$. Thus $\xi \preccurlyeq \eta^*$.

Note, that there exists a decomposition $\mathbf{1}_X = (x \wedge y) \oplus \xi \oplus \eta \oplus (x \vee y)^*$.

Let us apply this result in the case of $\mathbb{L}(\mathcal{H})$. Assume P and Q are compatible in $\mathbb{L}(\mathcal{H})$. One has the decompositions into orthogonal sums: $P = L \oplus U$, $Q = L \oplus V$, where $L = P \cap Q = \inf\{P, Q\}$. U and V satisfy $U \subset V^{\perp}$, i.e., $U \perp V$. It follows, that $[\widehat{\pi}_P, \widehat{\pi}_Q] = 0$. Thus the compatibility implies, that the corresponding orthogonal projectors commute.

On the other hand, if $[\widehat{\pi}_P, \widehat{\pi}_Q] = 0$, then $\mathcal{H}$ admits a decomposition into orthogonal sum $\mathcal{H} = L_0 \oplus L_1 \oplus L_2 \oplus L_3$, such, that $P = L_0 \oplus L_1$ and $Q = L_0 \oplus L_2$. From this one derives the two equalities required in the definition of compatible elements of $\mathbb{L}(\mathcal{H})$. Since any two elements of $for(\mathbb{L}(\mathcal{H}))$ have inf and sup (these will be intersection and the closure of a linear span), one deduces, that P and Q are compatible in the sense of the given definition. □

Note, that in the examples of $\mathbb{W}(\Omega, \mathcal{F})$ and $\mathbb{I}$, any two elements turn out to be compatible. For $\mathbb{W}(\Omega, \mathcal{F})$ the operations $\wedge$ and $\vee$ become $\cap$ and $\cup$ respectively, and the compatibility follows from the mutual distributivity of these operations. In case of $\mathbb{I}$ one has $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$, $x^* = 1 - x$ and the compatibility follows from the identity:

$$\min\{\xi, \eta\} + (1 - \xi) \equiv \eta + (1 - \max\{\xi, \eta\}), \qquad (\xi, \eta \in \mathbb{R}).$$

Proposition 3.4. *For every object $X = (D, E, \oplus)$ in $\mathcal{C}$ the following is true:*

$$\forall x, y \in D : x \preccurlyeq y \Rightarrow x \text{ is compatible with } y.$$

Proof: If $x \preccurlyeq y$, then $\inf\{x, y\} = x$ and $\sup\{x, y\} = y$. One has the equalities:

$$\inf\{x, y\} \oplus x^* = x \oplus x^* = \mathbf{1}_X = y \oplus y^* = y \oplus \sup\{x, y\}^*,$$
$$\inf\{x, y\} \oplus y^* = x \oplus y^* = x \oplus \sup\{x, y\}^*.$$

It follows, that x is compatible to y in X. □

Note, that from this proposition one gets, that $\mathbf{0}_X$ and $\mathbf{1}_X$ are compatible with any x.

We shall define now some subcategory $\mathcal{C}_1$ in $\mathcal{C}$. An object Y in $\mathcal{C}_1$ will have the properties, which allow to interpret $for(Y)$ as a set of binary measuring devices for an abstract physical system.

It is natural (for physical reasons), to think about the compatibility relation, as of a relation, which satisfies the following conditions:

- If x is compatible with y, then x is compatible with y^*.

- If x, y and z are pairwise compatible, then x is compatible with $y \wedge z$ and with $y \vee z$.

Let us formulate what is required more precisely. Let $X = (D, E, \oplus)$ be an object of $\mathcal{C}$. Denote

$$K := \{(x, y) \in D \times D | x \text{ is compatible with } y\},$$

$$\mathcal{P}_K(D) := \{T \in \mathcal{P}(D) | \forall x, y \in T : x \neq y \Rightarrow (x, y) \in K\}.$$

The set $\mathcal{P}_K(D)$ can be viewed as a partially ordered set with respect to inclusion of subsets. Denote by $\mathcal{M}_X$ the set of all its maximal elements: $\mathcal{M}_X := \text{Max}(\mathcal{P}_K(D); \subset)$. Consider the following conditions on X:

1) $\forall T \in \mathcal{P}_K(D) \exists M \in \mathcal{M}_X : M \supset T$.
2) $\forall M \in \mathcal{M}_X$: M is a Boolean lattice with respect to $(\wedge, \vee, *)$, where $\wedge$ and $\vee$ play the roles of AND and OR operations respectively, $*$ plays the role of NOT operation.

Note, that the first condition intuitively can be viewed as a sort of a condition, that the set of all maximal compatible sets covers the set D. Note, that $\mathbf{0}_X$ and $\mathbf{1}_X$ are compatible with any $x \in D$, and thus are present in any $M \in \mathcal{M}_X$. These two elements play the roles of minimal and maximal elements respectively for every maximal compatible set $M \in \mathcal{M}_X$.

One defines $\mathcal{C}_1$ as a full subcategory of $\mathcal{C}$ with objects satisfying these two conditions formulated above.

Proposition 3.5. *Let $X = (D, E, \oplus) \in \text{Ob}(\mathcal{C}_1)$. One claims, that any two elements $x, y \in D$, such that $x \preccurlyeq y^*$, are compatible and*

$$\inf\{x, y\} = \mathbf{0}_X, \qquad \sup\{x, y\} = x \oplus y.$$

Proof: Since $x \preccurlyeq y^*$, x is compatible with y^*. There exists a maximal compatible set, which contains x and y^*. This set is a Boolean lattice, and thus, in particular, it is invariant under the involution. It means, that x is compatible with $y^{**} = y$.

Now, since x and y^* are compatible, there exists a decomposition of $\mathbf{1}_X$, $\mathbf{1}_X = a \oplus \xi \oplus \eta \oplus b$, such that $a = \inf\{x, y^*\}$, $a \oplus \xi = x$, $a \oplus \eta = y$, $\eta \oplus b = x^*$, $\xi \oplus b = y^*$, $a \oplus \xi \oplus \eta = \sup\{x, y\}$. Let us show, that $a = \mathbf{0}_X$. (We do not write here the brackets using the associativity of $\oplus$). Indeed, $x \preccurlyeq y^*$ means, that $a \oplus \xi \preccurlyeq \xi \oplus b$, and this implies $a \preccurlyeq b$. At the same time, a and b are the terms of the $\oplus$-sum in the decomposition of $\mathbf{1}_X$, and this means they must satisfy $a \preccurlyeq b^*$, or, what is equivalent, $b \preccurlyeq a^*$. It follows, that $a \preccurlyeq b \preccurlyeq a^*$, and thus $a \preccurlyeq a^*$. It means, that $\inf\{a, a^*\} = a$. On the other hand, using one of the axioms of a Boolean lattice (the excluded middle), one obtains $\inf\{a, a^*\} = a \wedge a^* = \mathbf{0}_X$. It follows, that $a = \mathbf{0}_X$, i.e. $\inf\{x, y^*\} = \mathbf{0}_X$.

From this, one derives $x = a \oplus \xi = \mathbf{0}_X \oplus \xi = \xi$, and $y = a \oplus \eta = \mathbf{0}_X \oplus \eta = \eta$. It follows, that $\sup\{x, y\} = a \oplus \xi \oplus \eta = \mathbf{0}_X \oplus x \oplus y = x \oplus y$. □

Note, that since the formula $x \wedge x^* = \mathbf{0}_X$ is not true for $X = \mathbb{I}$, this object does not belong to the subcategory $\mathcal{C}_1$. The objects $\mathbb{W}(\Omega, \mathcal{F})$ and $\mathbb{L}(\mathcal{H})$ from the examples shown above happen to belong to $\mathcal{C}_1$.

We are now ready to formulate an abstract-hidden variables problem. Let $X \in \mathrm{Ob}(\mathcal{C}_1)$ and $f : X \to \mathbb{I}$ be a morphism of $\mathcal{C}$. The abstract-hidden variables problem for (X, f) is formulated as follows: construct a measurable space $(\Omega, \mathcal{F})$ such that there exists a monomorphism $\mu : X \rightarrowtail \mathbb{W}(\Omega, \mathcal{F})$; having constructed $((\Omega, \mathcal{F}), \mu)$, find a morphism $p : \mathbb{W}(\Omega, \mathcal{F}) \to \mathbb{I}$, such that $p \circ \mu = f$. This is illustrated by the following diagrams:

$$X \overset{\mu}{\dashrightarrow} \mathbb{W}(\Omega, \mathcal{F}) \tag{3}$$

$$\begin{array}{ccc} X & \overset{\mu}{\rightarrowtail} & \mathbb{W}(\Omega, \mathcal{F}) \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle p} & \\ \mathbb{I} & & \end{array} \tag{4}$$

Thus the abstract-hidden variables problem in fact splits into two subproblems. Note, that the formulation of the first subproblem (3) is similar to the notion of a cogenerating full subcategory, and the formulation of the second subproblem (4) reminds the definition of an injective object of a category.

The link between the just formulated problem and physics is established as follows. Recall that in the introduction we have introduced a notion of the expected relative frequency and a notation $(\mathfrak{D}, \mathfrak{R}, \Phi)$, where $\mathfrak{D}$ and $\mathfrak{R}$ are sets and Φ is a function $\Phi(\cdot|\cdot) : \mathfrak{D} \times \mathfrak{R} \to [0, 1]$. There exists a natural partial order on $\mathfrak{D}$: $P_1 \preccurlyeq P_2$ by definition iff $\forall S \in \mathfrak{R} : \Phi(P_1|S) \leq \Phi(P_2|S)$. One can also define some relation $\mathfrak{C}$ on $\mathfrak{D}$: $(P_1, P_2) \in \mathfrak{C}$ by definition iff $\forall S \in \mathfrak{R} : (\Phi(P_1|S) = 1 \Rightarrow \Phi(P_2|S) = 0) \& (\Phi(P_2|S) = 1 \Rightarrow \Phi(P_1|S) = 0)$. Assume that for every $(P_1, P_2) \in \mathfrak{C}$ there exists $\sup\{P_1, P_2\}$. It means, that one has a map $\cdot \oplus \cdot : \mathfrak{C} \to \mathfrak{D}, (P_1, P_2) \mapsto P_1 \oplus P_2 := \sup\{P_1, P_2\}$. One postulates that the data $(\mathfrak{D}, \mathfrak{C}, \oplus)$ defines an object of $\mathcal{C}_1$. Thus with every triple $(\mathfrak{D}, \mathfrak{R}, \Phi)$ one associates some $X \in \mathrm{Ob}(\mathcal{C}_1)$. Assuming that every element of $\mathfrak{D}$ has a representative which corresponds to an indication $(1) \in \mathcal{O}_{\{\mathbf{A}\}}$ of some measuring device $\mathbf{A} \in \mathcal{D}$ with $I_{\mathbf{A}} = \{0, 1\}$, one induces from the compatibility relation on $\mathcal{D}$ a relation on $\mathfrak{D}$. The latter is thought to be expressed by Definition 1. For every $S \in \mathfrak{R}$ one has a map $\Phi(\cdot|S) : \mathfrak{D} \to [0, 1]$. One postulates that this map defines a morphism $f_S : X \to \mathbb{I}$. It means that one may consider an abstract-hidden variables problem for every (X, f_S), $S \in \mathfrak{R}$. Suppose it has a solution $((\Omega_S, \mathcal{F}_S), \mu_S, p_S)$ for some $S \in \mathfrak{R}$. Then the elements of Ω_S are in fact what is called by physicists the *hidden variables*, since in practice one tries to find Ω_S as a subset of a real coordinate space. An element $\omega \in \Omega_S$ in this context is to be viewed as a *reason*, which *determines* the outcome of a measurement in a given experimental act (i.e. an indication $1 \in I_{\mathbf{A}} = \{0, 1\}$ *will* take place upon the measurement of $\mathbf{A}$ iff $for(\mu_S)(\mathbf{A}^{(1)}) \ni \omega$, where $\mathbf{A}^{(1)}$ is the element of $\mathfrak{D}$ associated to this indication).

It is worth mentioning that given a triple $(\mathfrak{D}, \mathfrak{R}, \Phi)$ one can define similar structures on $\mathfrak{R}$. In particular, there is a partial order on $\mathfrak{R}$: $S_1 \preccurlyeq S_2$ by definition iff $\forall P \in \mathfrak{D} : \Phi(P|S_1) \geq \Phi(P|S_2)$. Note that in quantum mechanics there exists a natural order-inversing injective map $\mathfrak{D} \rightarrowtail \mathfrak{R}$, $P \mapsto \min\{S \in \mathfrak{R} | \Phi(P|S) = 1\}$. The image $\mathfrak{M}$ of this map, being equipped with a partial order inherited from $\mathfrak{R}$, is in Galois duality with $(\mathfrak{D}; \preccurlyeq)$.

Definition 3.5. An object X of the category $\mathcal{C}_1$ is called *indeterministic*, if for any measurable space $(\Omega, \mathcal{F})$ there exists no monomorphism $\mu : X \rightarrowtail \mathbb{W}(\Omega, \mathcal{F})$.

In Section 2 we considered an example of a possible experimental data that leads to a contradiction when one tries to interpret the expected relative frequencies as probabilities. This may now be viewed as follows: there is an arrow $f_S : X \to \mathbb{I}$ defined by $\Phi(\cdot|S)$, $S \in \mathfrak{R}$, where X is an object of $\mathcal{C}_1$ associated to $(\mathfrak{D}, \mathfrak{R}, \Phi)$. The experimental data provides the knowledge about the restriction of $\Phi(\cdot|S)$ onto some finite subset of $\mathfrak{D}$. Specializing to the mentioned example, it is possible to say that even if one assumes the existence of $(\Omega_S, \mathcal{F}_S)$ and of a monomorphism μ_S, one cannot construct an arrow $p_S : \mathbb{W}(\Omega_S, \mathcal{F}_S) \to \mathbb{I}$ such that $p_S \circ \mu_S = f_S$.

Note, that in case of $\#D < \infty$, $D = for(X)$, one can derive a necessary and sufficient condition on the morphism f for the existence of p. It is necessary to consider all the linear inequalities of the form $0 \leq \overline{p}(\cap_{A \in U} \overline{\mu}(A)) \leq 1$, ($\overline{\mu} := for(\mu), \overline{p} := for(p)$), where U runs over *all* subsets of D. The quantities $\overline{p}(\cap_{A \in U} \overline{\mu}(A)) = \overline{f}(\wedge_{A \in U} A)$, $\overline{f} := for(f)$, should be interpreted as parameters whenever $U \in \mathcal{P}_K(D)$ (K denotes the compatibility relation on D), and as indeterminates otherwise. An explicit description of the set of all possible values of parameters, for which this system has a solution, gives the required necessary and sufficient condition. This type of condition was obtained in Accardi and Fedullo (1982). It turns out, that in quantum mechanics even the first subproblem in general does not have a solution and this is in fact the content of the Kochen–Specker theorem. It means, that indeterministic objects exist.

4. ATOMIC ELEMENTS

The object $X = \mathbb{L}(\mathbb{C}^n)$ of the category $\mathcal{C}_1$ has some additional properties in comparison to an abstract object X of $\mathcal{C}_1$. Let us state these properties.

Recall that $\mathcal{M}_X$ denotes the set $\mathcal{M}_X := \mathrm{Max}(\mathcal{P}_K(D); \subset)$, where $\mathcal{P}_K(D)$ is the set of subsets of pairwise compatible elements of $D := for(X)$. For every $M \in \mathcal{M}_X$, denote $B_X(M) := \mathrm{Min}(M^\times; \preccurlyeq)$, where $M^\times := M\backslash\{\mathbf{0}_X\}$. There exists a natural map $\rho_M : M \to \mathcal{P}(B_X(M))$,

$$\rho_M(x) := \{\xi \in B_X(M) | \xi \preccurlyeq x\}.$$

In case of $X = \mathbb{L}(\mathbb{C}^n)$, this map turns out to be a bijection, i.e. every element of M is *characterized by points*, i.e. by a subset of $B_X(M)$. Indeed, every $M \in \mathcal{M}_{\mathbb{L}(\mathbb{C}^n)}$ is determined by an orthonormal basis $e_1, e_2, \ldots, e_n$ in $\mathbb{C}^n$. M is formed by all elements of the form $\widehat{e}_I$, $I \subset \{1, 2, \ldots, n\}$, where

$$\widehat{e}_I := \mathrm{span}_{\mathbb{C}}\{e_i, i \in I\},$$

and, by definition, $\widehat{e}_\emptyset$ is a trivial subspace of $\mathbb{C}^n$. The elements of $B_{\mathbb{L}(\mathbb{C}^n)}(M)$ are the projective lines $\mathbb{C}e_i = \widehat{e}_{\{i\}}$, $i = \overline{1, n}$. Moreover, the operations $\wedge$, $\vee$, and $*$ correspond to the operations $\cap$, $\cup$, and $B_X(M)\backslash\cdot$, i.e., for every $x, y \in M$ one has

$$\rho_M(x \wedge y) = \rho_M(x) \cap \rho_M(y),$$
$$\rho_M(x \vee y) = \rho_M(x) \cup \rho_M(y),$$
$$\rho_M(x^*) = B_X(M)\backslash\rho_M(x).$$

Proposition 4.6. *Let $M \in \mathcal{M}_X$. One claims, that the assumption, that ρ_M is a bijection, implies, that*

$$\forall x \in M\, \forall \xi \in B_X(M) : \xi \preccurlyeq x \text{ or } x \preccurlyeq \xi^*.$$

Proof: If $\#for(X) = 1$, then $B_X(M) = \emptyset$ and the proposition is true. If $\#for(X) > 1$, then $B_X(M)$ can be represented as a union of two disjoint subsets: $B_X(M) = \rho_M(x) \cup \rho(x^*)$. If $\xi \in \rho_M(x)$, then $\xi \preccurlyeq x$. If $\xi \in \rho_M(x^*)$, then $\xi \preccurlyeq x^*$, what is equivalent to $x \preccurlyeq \xi^*$. □

Note, that from this proposition one concludes, that $\forall \xi, \eta \in B_X(M) : \xi \neq \eta \Rightarrow \xi \preccurlyeq \eta^*$. Indeed, $\xi \preccurlyeq \eta$ cannot be true for such ξ and η, since this due to the definition of $B_X(M)$ implies $\xi = \eta$. The possibility, that is left is $\xi \preccurlyeq \eta^*$.

Proposition 4.7. *For every $M, N \in \mathcal{M}_X$ and for any $\xi \in M \cap N$, if ξ happens to belong to $B_X(M)$, then ξ belongs to $B_X(N)$ as well.*

Proof: Let $\xi \in B_X(M) \cap N$. Take any $\eta \in \rho_N(\xi)$. It means, that $\eta \preccurlyeq \xi$, or, what is equivalent, $\xi^* \preccurlyeq \eta^*$. Using the previous proposition, for any $x \in M$, one obtains, that $\eta \preccurlyeq \xi \preccurlyeq x$ or $x \preccurlyeq \xi^* \preccurlyeq \eta^*$. In particular, it means, that η is compatible with x, i.e. $\eta \in M$. Since $\eta \preccurlyeq \xi$ and $\xi \in B_X(M)$, it follows, that $\eta = \xi$. Recalling, that $\eta \in B_X(N)$, one gets $\xi = \eta \in B_X(N)$. □

This fact makes the following notation reasonable:

$$\mathcal{A}_X := \{x \in for(X) | \exists M \in \mathcal{M}_X : B_X(M) \ni x\}.$$

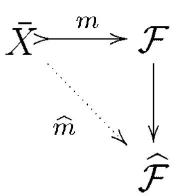
One may write $B_X(M)$ in the form $B_X(M) = \mathcal{A}_X \cap M$. We shall refer to the elements of $\mathcal{A}_X$ as *atomic* elements. Clearly in the case of $X = \mathbb{L}(\mathbb{C}^n)$ the set $\mathcal{A}_X$ is the set of all projective lines. Note, that for all $\xi, \eta \in \mathcal{A}_X$, if ξ happens to be compatible with η, then either ξ coincides with η, or $\xi \preccurlyeq \eta^*$.

5. SUFFICIENT CONDITION TO BE INDETERMINISTIC

We shall formulate now a sufficient condition for an object X of $\mathcal{C}_1$ to be indeterministic. Assume, that X is an object which admits a set of atomic elements $\mathcal{A}_X$. Thus, in particular, for any maximal compatible set $M \in \mathcal{M}_X$ the natural map $\rho_M : M \to \mathcal{P}(B_X(M))$, $B_X(M) = M \cap \mathcal{A}_X$, is a bijection. Assume, that $\forall M \in \mathcal{M}_X : \# B_X(M) < \infty$. Imagine that X admits the existence of a measurable space $(\Omega, \mathcal{F})$ and a monomorphism $\mu : X \rightarrowtail \mathbb{W}(\Omega, \mathcal{F})$. We describe first a procedure, which allows to simplify $(\Omega, \mathcal{F})$. It will be convenient to write $\bar{X}$ for $for(X)$ and m for $for(\mu)$. One defines an equivalence relation $\sim$ on the set Ω:

$$a \sim b \rightleftharpoons \forall x \in \bar{X} : (a, b \in m(x)) \,\text{or}\, (a, b \notin m(x)). \tag{5}$$

Note, that this sort of equivalence relation on a topological space was considered in Sorkin (1991). Note, that because of bijectivity of ρ_M and to finiteness of $B_X(M)$ (M runs over $\mathcal{M}_X$), the right hand side in the definition (5) is equivalent to $\forall \xi \in \mathcal{A}_X : (a, b \in m(\xi))$ or $(a, b \notin m(\xi))$. Thus, in order to establish $a \sim b$ it is sufficient to verify the condition of the definition only on atomic elements. Denote $\widehat{\Omega} = \Omega / \sim$, $\pi : \Omega \rightarrowtail \widehat{\Omega}$ – the natural projection. The map π induces a map $\mathcal{P}(\pi) : \mathcal{P}(\Omega) \to \mathcal{P}(\widehat{\Omega})$ between the power-sets, and a map $\mathcal{P}(\mathcal{P}(\pi)) : \mathcal{P}(\mathcal{P}(\Omega)) \to \mathcal{P}(\mathcal{P}(\widehat{\Omega}))$. Denote $\widehat{\mathcal{F}} := \mathcal{P}(\mathcal{P}(\pi))(\mathcal{F})$. One proves, that $(\widehat{\Omega}, \widehat{\mathcal{F}})$ is again a measurable space. Define a map $\widehat{m}$ from the requirement of commutativity of the diagram:



where the vertical arrow is induced by $\mathcal{P}(\pi)$. Claim, that $\widehat{m}$ in fact defines a morphism $\widehat{\mu} : X \to \mathbb{W}(\Omega, \mathcal{F})$, and moreover, this morphism happens to be a monomorphism. Thus, one gets another solution of the first part of the hidden variables problem for X with, perhaps, a more simple set $\widehat{\Omega}$ (i.e. a set of less cardinality).

Let us prove, that $\widehat{m}$ defines a morphism. For any $x, y \in \bar{X}$, one has

$$\begin{aligned}\widehat{m}(x \oplus y) &= \mathcal{P}(\pi)(m(x \oplus y)) = \mathcal{P}(\pi)(m(x) \sqcup m(y))\\ &= \{\pi(\omega)|\omega \in m(x) \sqcup m(y)\} = \{\pi(\omega)|\omega \in m(x)\} \sqcup \{\pi(\omega)|\omega \in m(y)\}\\ &= \mathcal{P}(\pi)(m(x)) \oplus \mathcal{P}(\pi)(m(y)) = \widehat{m}(x) \oplus \widehat{m}(y).\end{aligned}$$

For $m(\mathbf{1}_X)$ one has

$$\widehat{m}(\mathbf{1}_X) = \mathcal{P}(\pi)(m(\mathbf{1}_X)) = \mathcal{P}(\pi)(\Omega) = \{\pi(\omega)|\omega \in \Omega\} = \widehat{\Omega} = \mathbf{1}_{\mathbb{W}(\widehat{\Omega},\widehat{\mathcal{F}})}.$$

Thus, $\widehat{m}$ defines some morphism $\widehat{\mu} : X \to \mathbb{W}(\widehat{\Omega}, \widehat{\mathcal{F}})$. To prove, that this is in fact a monomorphism, one has to show, that $\widehat{m}$ is injective. Take any x, y, such that $x \neq y$. One has

$$\widehat{m}(x) = \{\pi(\omega)|\omega \in m(x)\},$$
$$\widehat{m}(y) = \{\pi(\omega)|\omega \in m(y)\}.$$

Since μ is a monomorphism, $m(x)$ and $m(y)$ are disjoint. One has to show, that $\widehat{m}(x)$ and $\widehat{m}(y)$ will be disjoint as well. Imagine the contrary: let $\widehat{m}(x) \cap \widehat{m}(y) \neq \emptyset$. Then there exist $\omega_1 \in m(x)$ and $\omega_2 \in m(y)$, such that $\pi(\omega_1) = \pi(\omega_2)$, i.e. $\omega_1 \sim \omega_2$. From the definition of the equivalence between the points of Ω, one obtains, that since $m(x) \ni \omega_1$, it contains both points ω_1 and ω_2, $m(x) \ni \omega_1, \omega_2$. But this, since $\omega_2 \in m(y)$, contradicts the disjointness of $m(x)$ and $m(y)$. This contradiction implies, that $\widehat{m}$ has to be injective, and it follows, that $\widehat{\mu}$ is a monomorphism.

We shall now establish an isomorphism between $\mathbb{W}(\widehat{\Omega}, \widehat{\mathcal{F}})$ and some other object of $\mathcal{C}$ of the form $\mathbb{W}(Q, \mathcal{F}_Q)$. Denote

$$\Gamma_X := \{s : \mathcal{M}_X \to \mathcal{A}_X | \forall M : s(M) \in B_X(M)\}$$

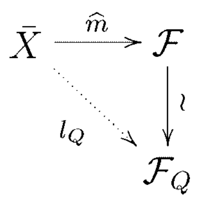
(Recall that the notation $\mathcal{A}_X$ was defined in the previous section and it means the set of all atomic elements.) Imagine, that one takes an arbitrary $s \in \Gamma_X$ and an arbitrary maximal-compatible subset $M \in \mathcal{M}_X$. Then one has $s(M) \in \mathcal{A}_X \subset \bar{X}$, and it is possible to apply a map $m : \bar{X} \to \mathcal{F}$. One gets $m(s(M)) \in \mathcal{F}$. We introduce now the following notations:

$$g_s^\mu := \bigcap\nolimits_{M \in \mathcal{M}_X} m(s(M)),$$
$$Q^\mu := \left\{s \in \Gamma_X \middle| g_s^\mu \neq \emptyset\right\}.$$

Note, that $Q^\mu \subset \Gamma_X$. We shall write Q instead of Q^μ not to overload the notations (thus μ is viewed as fixed). There exists a natural map $\varkappa : Q \to \widehat{\Omega}$, $s \mapsto \pi(\overset{\circ}{\omega})$, where $\overset{\circ}{\omega}$ is an arbitrary chosen element of $g_s^\mu \subset \Omega$. Let us prove that this map is well defined. One has to show, that any two points $\omega_1, \omega_2 \in g_s^\mu$ are equivalent, i.e., for all $\xi \in \mathcal{A}_X$ one must have either $\omega_1, \omega_2 \in m(\xi)$, or $\omega_1, \omega_2 \notin m(\xi)$. Take arbitrary

$\xi \in \mathcal{A}_X$. Then $\xi \in B_X(M)$, where M is some element of $\mathcal{M}_X$. Assume that one of the points, say ω_1, belongs to $m(\xi)$. Then $s(M) = \xi$, since otherwise $s(M) \preccurlyeq \xi^*$, what implies, that $m(s(M))$ and $m(\xi)$ are disjoint, implying a contradiction to $\omega_1 \in g_s^\mu \subset m(s(M))$. Thus $s(M) = \xi$ and $\omega_2 \in g_s^\mu \subset m(s(M)) = m(\xi)$. This proves, that any two points of g_s^μ are equivalent. Note now, that $\varkappa$ is not only well defined, but it is also, in fact, a bijection, $\varkappa : Q \xrightarrow{\sim} \widehat{\Omega}$. Indeed, for every $M \in \mathcal{M}_X$ one has a decomposition of Ω into a union of disjoint subsets (in such a case one writes Σ instead of $\cup$): $\Omega = \sum_{\xi \in B_X(M)} m(\xi)$. It follows, that $\Omega = \sum_{s \in \Gamma_X} g_s^\mu$. One can restrict the summation here only to $s \in Q$. It follows, that $\widehat{\Omega} = \mathcal{P}(\pi)(\Omega) = \sum_{s \in Q} \mathcal{P}(\pi)(g_s^\mu)$ (note, that any two points belonging to different g_s^μ sets are not equivalent). Since the map π sends all the points of a set g_s^μ into one and the same point in $\widehat{\Omega}$, $\mathcal{P}(\pi)(g_s^\mu)$ are the one-point sets of the form $\{\varkappa(s)\}$. Thus, $\widehat{\Omega} = \sum_{s \in Q}\{\varkappa(s)\}$. It follows, that $\varkappa$ establishes $Q \xrightarrow{\sim} \widehat{\Omega}$.

Since $\varkappa$ is a bijection, there exists $\varkappa^{-1}$, and one has the maps $\mathcal{P}(\varkappa^{-1}) : \mathcal{P}(\widehat{\Omega}) \to \mathcal{P}(Q)$ and $\mathcal{P}(\mathcal{P}(\varkappa^{-1})) : \mathcal{P}(\mathcal{P}(\widehat{\Omega})) \to \mathcal{P}(\mathcal{P}(Q))$. The latter map allows to define a σ-algebra on Q, $\mathcal{F}_Q := \mathcal{P}(\mathcal{P}(\varkappa^{-1}))(\widehat{\mathcal{F}})$. Thus, one gets a measurable space $(Q, \mathcal{F}_Q)$ and an object $\mathbb{W}(Q, \mathcal{F}_Q)$ of the category $\mathcal{C}$. Define a map l_Q from the commutative diagram:



where the vertical arrow is induced by $\mathcal{P}(\varkappa^{-1})$. It defines an isomorphism $\mathbb{W}(\Omega, \mathcal{F}) \xrightarrow{\sim} \mathbb{W}(Q, \mathcal{F}_Q)$. Since m defines a monomorphism $\mu : X \rightarrowtail \mathbb{W}(\Omega, \mathcal{F})$, the map l_Q will define some monomorphism λ_Q, $for(\lambda_Q) = l_Q$. It means that one gets a solution $\lambda_Q : X \rightarrowtail \mathbb{W}(Q, \mathcal{F}_Q)$ of the first subproblem of the hidden variables problem.

Proposition 5.8. *For any $X \in \mathrm{Ob}(\mathcal{C})$, $M \in \mathcal{M}_X$, $\xi \in B_X(M)$, the map $l_Q = for(\lambda_Q)$ acts according to the formula:*

$$for\,(\lambda_Q)(\xi) = \{s \in Q | s(M) = \xi\}.$$

Proof: According to the definitions,

$$l_Q(\xi) = \mathcal{P}(\varkappa^{-1})(\widehat{m}(\xi)) = \mathcal{P}(\varkappa^{-1})(\mathcal{P}(\pi)(m(\xi))).$$

Expressing $m(\xi)$ as a union of disjoint sets $\sum_{s\in Q:s(M)=\xi} g_s^{\mu}$, one deduces

$$l_Q(\xi) = \mathcal{P}(\varkappa^{-1})\left(\left\{\pi(\omega)|\omega \in \sum_{s\in Q:s(M)=\xi} g_s^{\mu}\right\}\right)$$

$$= \mathcal{P}(\varkappa^{-1})\left(\sum_{s\in Q:s(M)=\xi} \{\pi(\omega)|\omega \in g_s^{\mu}\}\right).$$

Noting, that in the latter sum the expression $\pi(\omega)$ may be replaced by $\varkappa(s)$, one deduces:

$$l_Q(\xi) = \mathcal{P}(\varkappa^{-1})(\mathcal{P}(\varkappa)(\{s \in Q|s(M) = \xi\})) = \{s \in Q|s(M) = \xi\}.$$

□

Note, that it may happen that for some $\xi \in \mathcal{A}_X$ one has $\xi \in M \cap N$ for some $M, N \in \mathcal{M}_X$. In this case one must have an equality

$$\{s \in Q|s(M) = \xi\} = \{s \in Q|s(N) = \xi\},$$

which can be viewed as a condition on the set Q.

Note, that for an arbitrary element a the following formula is true:

$$for(\lambda_Q)(a) = \{s \in Q|s(M) \preccurlyeq a\},$$

where $M \in \mathcal{M}_X$ is any subset containing a. Indeed, after having chosen $M \ni a$, one can uniquely express a in the form $a = \oplus_{\xi\in B_X(M),\xi\, \preccurlyeq\, a}\xi$. Then the mentioned formula follows from the bijectivity of the map ρ_M and the fact, that $for(\lambda_Q)$ sends $\oplus$-sums into disjoint unions.

Note the following. If $s \in Q$, then for any $M, N \in \mathcal{M}_X$ it is impossible to have $s(M) \preccurlyeq s(N)^*$. Indeed, if this were the case, the sets $m(s(M))$ and $m(s(N))$ have to be disjoint, what implies, that $g_s^{\mu} = \emptyset$ and thus $s \notin Q$. Denote

$$D_X := \{s \in \Gamma_X|\forall M, N : \neg(s(M) \preccurlyeq s(N)^*)\}.$$

One has: $Q \subset D_X$. Recall, that Q in fact depends on μ, and one just omits μ in the notation ($Q = Q^{\mu}$). At the same time, D_X is defined without reference to μ. Note that for any $Q_1 \subset D_X$ one has the coincidence $\{s \in Q_1 \,|\, s(M) = \xi\} = \{s \in Q_1|s(N) = \xi\}$, where ξ is an arbitrary atomic element from $M \cap N$, $M, N \in \mathcal{M}_X$. Indeed, take any $s \in Q_1$ such that $s(M) = \xi$. Then one has: $\xi, s(N) \in B_X(N)\,\&\,\neg(s(N) \preccurlyeq \xi^*)$. This implies that $s(N) = \xi$ and thus the coincidence mentioned does take place.

Since $Q \subset D_X$, one immediately deduces a sufficient condition for the solution of the first part of the hidden-variables problem not to exist: this is $D_X = \emptyset$. In practice, in order to show that D_X is empty, it might be more convenient to try to do the following. For an arbitrary $U \subset \mathcal{M}_X$ denote

$$\Gamma_X(U) := \{s : U \to \mathcal{A}_X | \forall M : s(M) \in B_X(M)\},$$
$$D_X(U) := \{t \in \Gamma_X(U) | \forall M, N \in U : \neg(t(M) \preccurlyeq t(N)^*)\}.$$

Note, that $D_X = D_X(\mathcal{M}_X)$. It follows from the formula

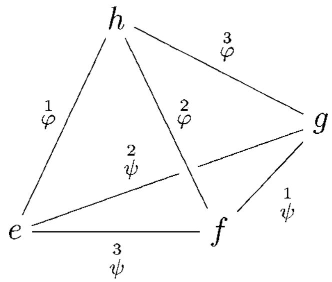
$$D_X(U_1 \cup U_2) = \{t \in \Gamma_X(U_1 \cup U_2)|$$
$$1) t|_{U_1} \in D_X(U_1), \quad 2) t|_{U_2} \in D_X(U_2),$$
$$3) \forall M_1 \in U_1 \, \forall M_2 \in U_2 : \neg(t(M_1) \preccurlyeq t(M_2)^*)\},$$

that in order to show that $D_X = \emptyset$, it is sufficient to show that $D_X(U) = \emptyset$ for some U. Thus we obtain the following theorem:

Theorem 5.1. *Let X be an object of $\mathcal{C}_1$, such that for every $M \in \mathcal{M}_X$ the number of elements in $B_X(M)$ is finite and the natural map $\rho_M : M \to \mathcal{P}(B_X(M))$ is bijective. If there exists $U \subset \mathcal{M}_X$ such that $D_X(U) = \emptyset$, then this object is indeterministic.*

6. AN EXAMPLE OF AN INDETERMINISTIC OBJECT

We shall construct now an example of the case when the situation $D_X(U) = \emptyset$ described in the previous section occurs. Namely, the object X will be of the form $X = \mathbb{L}(\mathcal{H})$, $\mathcal{H} \simeq \mathbb{C}^n$ (n is some natural number), and we shall describe a *finite* set U of maximal compatible subsets, such that $D_X(U) = \emptyset$. The general idea of this example is induced by Mermin (1993), although now the corresponding construction comes from a different context. Note, that the results of the mentioned work were also used in Kernaghan and Peres (1995) to derive a simple proof of "no-hidden variables" theorem in nonrelativistic quantum mechanics. We shall need some auxiliary construction. Let G denote a group, which is a direct product of several (say, m) copies of $\mathbb{Z}_2$: $G := \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ (m times). Let us view $\mathbb{Z}_2$ as a group consisting of 0 and 1 with a group operation written additively (addition modulo 2). Consider a graph



(6)

This graph is a completely connected nonoriented graph with 4 vertices. The vertices are denoted as e, f, g, and h. Denote the set of vertices as V and the set of edges as $E := \{\epsilon \in \mathcal{P}(V) | \#\epsilon = 2\}$. The zero element of G will be denoted as $\mathbf{0} := (0, 0, \ldots, 0)$ and $\mathbf{g}$ will denote an element $\mathbf{g} := (1, 1, \ldots, 1)$. Every label of an edge on this graph is unique and is of the form $\overset{\omega}{\varphi}$ or $\overset{\omega}{\psi}$, where ω is one of the numbers 1, 2, or 3. So far φ and ψ are just two formal symbols. Note, that for every vertex the edges, those are associated to it, have different numbers. This allows to assign to every vertex the following formal expressions: $e \mapsto \varphi \otimes \psi \otimes \psi$, $f \mapsto \psi \otimes \varphi \otimes \psi$, $g \mapsto \psi \otimes \psi \otimes \varphi$, $h \mapsto \varphi \otimes \varphi \otimes \varphi$. The general rule should be clear from the figure of the graph (rf. (6)): one takes a vertex and considers all the edges associated to it. The expression with $\otimes\otimes$ is constructed out of the symbols, assigned to the edges, and the numbers are interpreted as the positions of the corresponding symbols.

Let us take now an arbitrary orthonormal basis $\{\varphi_\alpha\}_\alpha$ in $\mathbb{C}^{2^m}$ indexed by the elements of the group G, $\alpha \in G$. Define a map $v : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{C}^\times$, $v(x, y) = -1$ if $(x, y) = (1, 1)$ and $v(x, y) = 1$ otherwise. This map has the following properties: $v(x, y) = v(y, x)$, $v(x, y + z) = v(x, y)v(x, z)$, $v(x, 1 + x) \equiv 1$, $\sum_x v(x, y)v(x, z) = 2\,\delta_{y,z}$ (here x, y and z are variables running over $\mathbb{Z}_2$). For every $\alpha, \beta \in G$, $\alpha = (\alpha_1, \ldots, \alpha_m)$, $\beta = (\beta_1, \ldots, \beta_m)$, denote $u_\alpha^\beta := \frac{1}{2^{m/2}} v(\alpha_1, \beta_1) \cdots v(\alpha_m, \beta_m)$. Thus, one gets a matrix $u = \|u_\alpha^\beta\|_{\alpha,\beta}$ with the following properties:

$$
\begin{aligned}
u_\alpha^\beta &= u_\beta^\alpha, \\
u_{\alpha+\beta}^\xi u_\beta^\eta &= u_\alpha^\xi u_\beta^{\xi+\eta}, \\
u_\alpha^{\beta+\alpha} &= u_\alpha^{\beta+\mathbf{g}}, \\
\sum_{\alpha\in G} u_\alpha^\beta u_\alpha^\gamma &= \delta_{\beta,\gamma},
\end{aligned}
$$

where α, β, γ, ξ, and η run over G. In particular, the latter formula allows to define another orthonormal basis $\{\psi_\lambda\}_{\lambda\in G}$ in $\mathbb{C}^{2^m}$ by the formula: $\psi_\lambda = \sum_{\alpha\in G} u_\lambda^\alpha \varphi_\alpha$. Note, that $\varphi_\alpha = \sum_{\lambda\in G} u_\alpha^\lambda \psi_\lambda$.

We shall describe now some elements in $\mathcal{H} := (\mathbb{C}^{2^m})^{\otimes 3}$. Denote $E_v := \{\epsilon \in E | \epsilon \ni v\}$, where $v \in V$. With every vertex $v \in V$ and every function $\sigma : E_v \to G$, one associates a function $\Psi_{\sigma(\cdot)}^v \in \mathcal{H}$. It is best to illustrate the general formula for $\Psi_{\sigma(\cdot)}^v$ by an example with $v = e$. Recall that one has assigned to this vertex a formal expression of the form $\varphi \otimes \psi \otimes \psi$. Take $\sigma : E_e \to G$. The function $\Psi_{\sigma(\cdot)}^e$ is defined by the formula

$$
\Psi_{\sigma(\cdot)}^e := \varphi_{\sigma(eh)} \otimes \psi_{\sigma(eg)} \otimes \psi_{\sigma(ef)},
$$

where one writes for short eh, eg, and ef instead of $\{e, h\}$, $\{e, g\}$, and $\{e, f\}$ respectively to denote the edges associated to the vertex e (note that eh has the number 1, eg has the number 2 and ef has the number 3). Let us point out, that the function $\Psi^e_{\sigma(\cdot)}$ has a unit norm and that the functions $\Psi^e_{\sigma(\cdot)}$ and $\Psi^e_{\sigma'(\cdot)}$ corresponding to different functions σ and σ', are orthogonal. The functions, corresponding to the other three vertices, are defined in a similar way. Note, that $\Psi^v_{\sigma(\cdot)}$ and $\Psi^u_{\sigma_1(\cdot)}$ corresponding to different $u, v \in V$, ($\sigma : E_v \to G$, $\sigma_1 : E_u \to G$), will be orthogonal iff $\sigma(uv) \neq \sigma_1(uv)$. Denote the set of all maps from E_v to G as F_v. Since $\#F_v = (2^m)^3 = \dim\mathcal{H}$, one gets an orthonormal basis $\{\Psi^v_{\sigma(\cdot)}\}_{\sigma\in F_v}$ in $\mathcal{H}$ defined for every vertex $v \in V$. Denote by M_v the maximal compatible set in $X = \mathbb{L}(\mathcal{H})$, corresponding to the basis $\{\Psi^v_{\sigma(\cdot)}\}_{\sigma\in F_v}$, i.e. $B_X(M_v) = \{\mathbb{C}\Psi^v_{\sigma(\cdot)}\}_{\sigma\in F_v}$ (recall that the atomic elements in $\mathbb{L}(\mathcal{H})$ are projective lines in $\mathcal{H}$).

We shall now define another orthonormal basis in $\mathcal{H}$ and then introduce a maximal compatible set $\widehat{M}$ associated to it. After that it will be shown, that for $U_0 := \sqcup_{v\in V}\{M_v\}$ one has $D_X(U_0) \neq \emptyset$, but for $U := U_0 \sqcup \{\widehat{M}\}$ one has $D_X(U) = \emptyset$ ($X = \mathbb{L}(\mathcal{H})$). Denote

$$\Lambda := \left\{\pi : V \to G | \sum_{v\in V} \pi(v) = \mathbf{g}\right\}.$$

Thus, if one has a $\pi \in \Lambda$ and knows its values on three of the vertices, the value on the fourth vertex is automatically determined. It follows that $\#\Lambda = (2^m)^3 = \dim\mathcal{H}$. We shall construct a set of pairwise orthogonal projective lines in $\mathcal{H}$ indexed by elements of Λ. Let us take an *ordered* pair of vertices (u, v), $u \neq v$. Denote the other two elements of V as z and w. Associate to every such pair (u, v) and every $\pi \in \Lambda$ a function

$$F^{(u\to v)}_{\pi(\cdot)} := \sum_{\sigma(\cdot)\in F_u} A_{\pi(z),\pi(w)}(\sigma(uz), \sigma(uw))\delta_{\sigma(uv)+\sigma(uz)+\sigma(uw),\pi(u)}\Psi^u_{\sigma(\cdot)}, \tag{7}$$

where $A_{x,y}(\xi, \eta) := u^{y+\eta}_x u^{y+\eta}_\xi$. Note, that the expression $A_{x,y}(\xi, \eta)$ is symmetric with respect to permutation $(x, \xi) \rightleftarrows (y, \eta)$. Indeed, using the properties of u, one deduces:

$$A_{x,y}(\xi, \eta) = u^{y+\eta}_x u^{y+\eta}_\xi = u^{y+\eta}_{(x+\xi)+\xi} u^{y+\eta}_\xi = u^{y+\eta}_{x+\xi} u^{\mathbf{0}}_\xi.$$

Noting that, $u^{\xi+\eta}_{\mathbf{0}} u^{\xi}_{\mathbf{0}} = u^{\xi+\eta}_{\mathbf{0}+\mathbf{0}} u^{\xi}_{\mathbf{0}} = u^{\xi+\eta}_{\mathbf{0}} u^{(\xi+\eta)+\xi}_{\mathbf{0}} = u^{\xi+\eta}_{\mathbf{0}} u^{\eta}_{\mathbf{0}}$ and taking into account, that by definition all the elements of u are not equal to zero, one gets $u^{\xi}_{\mathbf{0}} = u^{\eta}_{\mathbf{0}}$. It follows, that

$$A_{x,y}(\xi, \eta) = u^{x+\xi}_{y+\eta} u^{\mathbf{0}}_\eta = A_{y,x}(\eta, \xi).$$

Proposition 6.9.

1) *The projective line* $\mathbb{C}F^{(u\to v)}_{\pi(\cdot)}$ *does not depend upon the choice of the ordered pair* (u, v).

2) $\mathbb{C}F^{(u\to v)}_{\pi(\cdot)} \perp \mathbb{C}F^{(u_1\to v_1)}_{\pi_1(\cdot)}$ *iff* $\pi(\cdot) \neq \pi_1(\cdot)$.
3) $\mathbb{C}F^{(u\to v)}_{\pi(\cdot)} \perp \mathbb{C}\Psi^{w}_{\sigma(\cdot)}$ *iff* $\sum_{\epsilon\in E_w} \sigma(\epsilon) \neq \pi(w)$.

Proof:

1) Take arbitrary $\pi \in \Lambda$. Denote $\pi(e) = a, \pi(f) = b, \pi(g) = c$ and $\pi(h) = d$. Note, that $a + b + c + d = \mathbf{g}$. We shall show, that $\mathbb{C}F^{(h\to g)}_{\pi(\cdot)} = \mathbb{C}F^{(h\to f)}_{\pi(\cdot)} = \mathbb{C}F^{(f\to h)}_{\pi(\cdot)}$. The rest is similar. The functions $F^{(h\to g)}_{\pi(\cdot)}$, $F^{(h\to f)}_{\pi(\cdot)}$ and $F^{(f\to h)}_{\pi(\cdot)}$ defined by (7) are of the form:

$$F^{(h\to g)}_{\pi(\cdot)} = \sum_{\alpha,\beta\in G} u_a^{b+\beta} u_\alpha^{b+\beta} \varphi_\alpha \otimes \varphi_\beta \otimes \varphi_{d+\alpha+\beta},$$

$$F^{(h\to f)}_{\pi(\cdot)} = \sum_{\alpha,\gamma} u_a^{c+\gamma} u_\alpha^{c+\gamma}\, \varphi_\alpha \otimes \varphi_{d+\alpha+\gamma} \otimes \varphi_\gamma,$$

$$F^{(f\to h)}_{\pi(\cdot)} = \sum_{\xi,\zeta} u_c^{a+\zeta} u_\xi^{a+\zeta}\, \psi_\xi \otimes \varphi_{b+\xi+\zeta} \otimes \psi_\zeta.$$

Consider $F^{(h\to g)}_{\pi(\cdot)}$. Introducing a new index for summation $\gamma = d + \alpha + \beta$ and expressing β as $\beta = d + \alpha + \gamma$ (recall, that $\forall \xi \in G : \xi + \xi = \mathbf{0}$), one deduces

$$F^{(h\to g)}_{\pi(\cdot)} = \sum_{\alpha,\gamma} u_a^{b+(d+\alpha+\gamma)} u_\alpha^{b+(d+\alpha+\gamma)}\, \varphi_\alpha \otimes \varphi_{d+\alpha+\gamma} \otimes \varphi_\gamma.$$

Taking into account that $b + d = \mathbf{g} + a + c$ and using the properties of the matrix u, one deduces:

$$\begin{aligned} u_a^{b+d+\alpha+\gamma} u_\alpha^{b+d+\alpha+\gamma} &= u_a^{\mathbf{g}+a+c+\alpha+\gamma} u_\alpha^{\mathbf{g}+a+c+\alpha+\gamma} \\ &= u_a^{\mathbf{g}+\mathbf{g}+c+\alpha+\gamma} u_\alpha^{\mathbf{g}+a+c+\mathbf{g}+\gamma} = u_a^{c+\alpha+\gamma} u_\alpha^{a+c+\gamma} \\ &= u^a_{(c+\gamma)+\alpha} u_\alpha^{a+c+\gamma} = u^a_{c+\gamma} u_\alpha^{a+(a+c+\gamma)} \\ &= u^a_{c+\gamma} u_\alpha^{c+\gamma} = u_a^{c+\gamma} u_\alpha^{c+\gamma}. \end{aligned}$$

It follows, that $\mathbb{C}F^{(h\to g)}_{\pi(\cdot)}$ coincides with $\mathbb{C}F^{(h\to f)}_{\pi(\cdot)}$.

Let us show now, that $\mathbb{C}F^{(h\to g)}_{\pi(\cdot)} = \mathbb{C}F^{(f\to h)}_{\pi(\cdot)}$. Substituting $\varphi_\alpha = \sum_\xi u_\alpha^\xi \psi_\xi$ and $\varphi_{d+\alpha+\beta} = \sum_\zeta u^\zeta_{d+\alpha+\beta} \psi_\zeta$ in the corresponding expression for $F^{(h\to g)}_{\pi(\cdot)}$, one obtains

$$F^{(h\to g)}_{\pi(\cdot)} = \sum_{\alpha,\beta,\xi,\zeta} u_a^{b+\beta} u_\alpha^{b+\beta}\, u_\alpha^\xi u^\zeta_{d+\alpha+\beta}\, \psi_\xi \otimes \varphi_\beta \otimes \psi_\zeta.$$

Using the properties of u, one deduces $u_\alpha^\xi u^\zeta_{d+\alpha+\beta} = u^\zeta_{(d+\beta)+\alpha} u_\alpha^\xi = u^\zeta_{d+\beta} u_\alpha^{\zeta+\xi}$. Substituting this expression into the previous formula and performing the summation over α, one gets

$$F_{\pi(\cdot)}^{(h\to g)} = \sum_{\beta,\xi,\zeta} u_a^{b+\beta} u_{d+\beta}^{\zeta} \delta_{b+\beta,\zeta+\xi} \psi_\xi \otimes \varphi_\beta \otimes \psi_\zeta$$

$$= \sum_{\xi,\zeta} u_a^{\xi+\zeta} u_{\mathbf{g}+a+c+\xi+\zeta}^{\zeta} \psi_\xi \otimes \varphi_{b+\xi+\zeta} \otimes \psi_\zeta .$$

Using the properties of u, one may transform the coefficients in the terms of latter sum as follows:

$$u_a^{\xi+\zeta} u_{\mathbf{g}+a+c+\xi+\zeta}^{\zeta} = u_a^{\xi+\zeta} u_{\mathbf{g}+a+c+\xi+\mathbf{g}}^{\zeta}$$

$$= u_{(c+\xi)+a}^{\zeta} u_a^{\xi+\zeta} = u_{c+\xi}^{\zeta} u_a^{\xi} = u_{c+\xi}^{\zeta} u_\xi^{a} = u_c^{\zeta} u_\xi^{\zeta+a}.$$

Note, that $u_c^a u_c^\zeta = u_c^{(a+\zeta)+\zeta} u_c^\zeta = u_c^{a+\zeta} u_{c+c}^{\zeta} = u_c^{a+\zeta} u_{\mathbf{0}}^{\zeta}$. It means, that $u_c^\zeta u_\xi^{\zeta+a} = u_c^{a+\zeta} u_{\mathbf{0}}^{\zeta} u_\xi^{a+\zeta}/u_c^a$. Since $u_{\mathbf{0}}^{\zeta}$ is a constant, one concludes, that $\mathbb{C}F_{\pi(\cdot)}^{(h\to g)} = \mathbb{C}F_{\pi(\cdot)}^{(f\to h)}$.

2) Let us calculate the inner product of $F_{\pi(\cdot)}^{(h\to g)}$ and $F_{\pi_1(\cdot)}^{(h\to g)}$. Denote $a_1 = \pi_1(e)$, $b_1 = \pi_1(f)$, $c_1 = \pi_1(g)$, $d_1 = \pi_1(h)$. Note, that $a_1 + b_1 + c_1 + d_1 = \mathbf{g}$. Taking into account, that $(\varphi_\alpha, \varphi_\beta) = \delta_{\alpha,\beta}$, one deduces:

$$\left(F_{\pi(\cdot)}^{(h\to g)}, F_{\pi_1(\cdot)}^{(h\to g)}\right) = \sum_{\alpha,\beta,\alpha_1,\beta_1} \overset{*}{u}{}_a^{b+\beta} \overset{*}{u}{}_\alpha^{b+\beta} u_{a_1}^{b_1+\beta_1} u_{\alpha_1}^{b_1+\beta_1} \times$$

$$\delta_{\alpha,\alpha_1}\delta_{\beta,\beta_1}\delta_{d+\alpha+\beta,d_1+\alpha_1+\beta_1} = \sum_{\alpha,\beta} \overset{*}{u}{}_a^{b+\beta} \overset{*}{u}{}_\alpha^{b+\beta} u_{a_1}^{b_1+\beta} u_\alpha^{b_1+\beta} \delta_{d+\alpha+\beta,d_1+\alpha+\beta}$$

$$= \sum_{\beta} \overset{*}{u}{}_a^{b+\beta} u_{a_1}^{b_1+\beta} \delta_{b+\beta,b_1+\beta}\delta_{d,d_1}$$

$$= \delta_{b,b_1}\delta_{d,d_1} \sum_{\beta} \overset{*}{u}{}_a^{b+\beta} u_{a_1}^{b+\beta} = \delta_{a,a_1}\delta_{b,b_1}\delta_{d,d_1}.$$

3) We show that $\mathbb{C}\Psi_{\sigma(\cdot)}^{h} \perp \mathbb{C}F_{\pi(\cdot)}^{(u\to v)}$. The rest is similar. Let us calculate the inner product of $F_{\pi(\cdot)}^{(h\to g)}$ and $\Psi_{\sigma(\cdot)}^{h}$:

$$\left(F_{\pi(\cdot)}^{(h\to g)}, \Psi_{\sigma(\cdot)}^{h}\right) = \sum_{\alpha,\beta} u_{\pi(e)}^{\pi(f)+\beta} u_\alpha^{\pi(f)+\beta} \delta_{\alpha,\sigma(he)}\delta_{\beta,\sigma(hf)}\delta_{\pi(h)+\alpha+\beta,\sigma(hg)}$$

$$= u_{\pi(e)}^{\pi(f)+\sigma(hf)} u_{\sigma(he)}^{\pi(g)+\sigma(hf)} \delta_{\pi(h)+\sigma(he)+\sigma(hf),\sigma(hg)}.$$

Since u_ξ^η is never zero, the statement follows. □

The latter proposition means, that one has a set of $n = \dim\mathcal{H}$ pairwise orthogonal projective lines in $\mathcal{H}$. Denote the corresponding maximal compatible set in $X = \mathbb{L}(\mathcal{H})$ as $\widehat{M}$. Recall, that $U_0 := \sqcup_{v\in V}\{M_v\}$ and $U := U_0 \sqcup \{\widehat{M}\}$. Let us show, that $D_X(U_0) \neq \emptyset$, but $D_X(U) = \emptyset$. Denote by Func(E, G) the set of all maps from E to G. There is a natural map χ : Func$(E, G) \to D_X(U_0)$, $\tau \mapsto t_\tau$, $t_\tau(M_v) = \mathbb{C}\Psi^v_{\tau|_{E_v}}$. This map is in fact a bijection. Indeed, take any $s \in D_X(U_0)$. The values $s(M_u)$ and $s(M_v)$ are of the form $s(M_u) = \mathbb{C}\Psi^u_{\sigma(\cdot)}$, $s(M_v) = \mathbb{C}\Psi^v_{\sigma_1(\cdot)}$, where σ and σ_1 are some elements of F_u and F_v respectively. One has $\neg(s(M_v) \preccurlyeq s(M_u)^*)$. This reduces to $\neg(\mathbb{C}\Psi^u_{\sigma(\cdot)} \perp \mathbb{C}\Psi_{\sigma_1(\cdot)})$, i.e. $\sigma(uv) = \sigma_1(uv)$. This allows to define a map $\widetilde{\chi} : D_X(U_0) \to$ Func(E, G) by $\widetilde{\chi}(s)(uv) := \sigma(uv) = \sigma_1(uv)$. Then $\widetilde{\chi}$ will be an inverse map to χ. In particular, since Func$(E, G) \neq \emptyset$, it means that $D_X(U_0) \neq \emptyset$.

Let us now show, that $D_X(U_0 \cup \{\widehat{M}\}) = \emptyset$. $D_X\big(U_0 \cup \{\widehat{M}\}\big)$ should consist of all functions defined on $U_0 \cup \{\widehat{M}\}$ with values in $\mathcal{A}_X$, which satisfy the conditions: $t|_{U_0} \in D_X(U_0)$ and $\forall v \in V : \neg(t(M_v) \preccurlyeq t(\widehat{M})^*)$. Imagine, that $D_X(U_0 \cup \{\widehat{M}\}) \neq \emptyset$. Then one can take $t \in D_X\big(U_0 \cup \{\widehat{M}\}\big)$. Set $\tau = \chi^{-1}(t|_{U_0})$ and define $\pi(\cdot)$ from $t(\widehat{M}) = \mathbb{C}F^{(u\to v)}_{\pi(\cdot)}$ (u, v are any elements of V, $u \neq v$). From the proposition it follows that

$$\forall v \in V : \quad \pi(v) = \sum_{\epsilon\in E_v} \tau|_{E_v}(\epsilon).$$

Take the sum over all $v \in V$. From the definition of Λ one has $\sum_{v\in V} \pi(v) = \mathbf{g}$. On the other hand,

$$\sum_{v\in V} \pi(v) = \sum_{v\in V}\sum_{\epsilon\in E_v} \tau|_{E_v}(\epsilon) = 2\sum_{\epsilon\in E} \tau(\epsilon).$$

Taking into account that $\forall x \in G : x + x = \mathbf{0} \equiv (0, 0, \ldots, 0)$, one arrives at a contradiction of the form $\mathbf{g} = \mathbf{0}$. Hence: $D_X(U) = \emptyset$. It follows, that $X = \mathbb{L}(\mathcal{H})$ satisfies the sufficient condition for an indeterministic object formulated in the theorem.

7. SEMICLASSICAL CASE

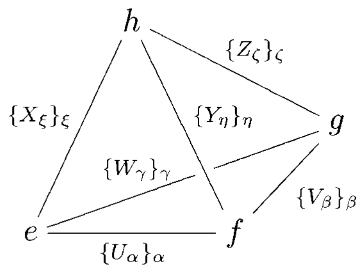
Let us analyze now the passage to the semiclassical limit in the context of the hidden variables problem. Consider the example of the previous section. Assume for simplicity, that $G = \mathbb{Z}_2$ and thus $\mathcal{H} = \mathbb{C}^8$. In this case $\#\Lambda = 8$ and $\forall v \in V$: $\#F_v = 8$. One has five orthonormal bases in $\mathbb{C}^8$: $\{F_{\pi(\cdot)}\}_{\pi\in\Lambda}$ and $\{\Psi^v_{\sigma(\cdot)}\}_{\sigma\in F_v}$, ($v \in V$, $\#V = 4$), where $F_{\pi(\cdot)} \equiv F^{(h\to g)}_{\pi(\cdot)}$. It was shown that these data induce a sufficient condition formulated in the theorem for the object $\mathbb{L}(\mathcal{H})$ to be indeterministic. This fact may be viewed as follows. Choose four functions $\sigma_v \in F_v$, $v \in V$, and assume, that for all $u, v \in V$, $u \neq v$, the function $\Psi^u_{\sigma_u(\cdot)}$ *is not* orthogonal to $\Psi^v_{\sigma_v(\cdot)}$. Then every function $F_{\pi(\cdot)}$ is orthogonal to at least one of $\Psi^v_{\sigma_v(\cdot)}$.

Let us analyze this fact in a more general way. Consider a Hilbert space $\mathbb{C}^n$, $n \in \mathbb{N}$. There are four orthonormal bases $\{e_i\}^n_{i=1}$, $\{f_j\}^n_{j=1}$, $\{g_k\}^n_{k=1}$ and $\{h_l\}^n_{l=1}$. The set of points

$$\Delta := \{(i, j, k, l) | \{e_i, f_j, g_k, h_l \text{ are pairwise nonorthogonal}\}$$

is not empty. Then, the situation one encounters, in particular, means the following: there exists a nonzero element $\Phi \in \mathbb{C}^n$, such that for any $(i, j, k, l) \in \Delta$ the vector Φ happens to be orthogonal to at least one of the four vectors e_i, f_j, g_k, or h_l.

Consider a completely connected nonoriented graph with four vertices denoted by symbols e, f, g, and h. We shall associate to every edge some decomposition of $\mathbb{C}^n$ into orthogonal sum of subspaces. Let us illustrate this for the edge ef. Denote $\overline{n} := \{1, 2, \ldots, n\}$. Let I and J be some subsets in $\overline{n}$. One calls the pair (I, J) *coherent*, if $\widehat{e_I} = \widehat{f_J}$. One calls a coherent pair (I, J) *irreducible*, if $I_1 \subset I \& J_1 \subset J \& \widehat{e}_{I_1} = \widehat{f}_{J_1}$ implies $I_1 = I \;\&\; J_1 = J$ or $I_1 = \emptyset \;\&\; J_1 = \emptyset$. Note, that if (I, J) is coherent, then $(\overline{n} \backslash I, \overline{n} \backslash J)$ is also coherent. Note, that if (I_1, J_1) and (I_2, J_2) are two irreducible coherent pairs, then there are only two possibilities: either $I_1 = I_2 \& J_1 = J_2$, or $I_1 \cap I_2 = \emptyset \& J_1 \cap J_2 = \emptyset$. This follows from the formulas $\widehat{e}_{I_1} \cap \widehat{e}_{I_2} = \widehat{e}_{I_1 \cap I_2}$, $\widehat{f}_{J_1} \cap \widehat{f}_{J_2} = \widehat{f}_{J_1 \cap J_2}$. It follows, that there exists a uniquely defined family $\{(I_\alpha, J_\alpha)\}_\alpha$, α runs over some index set A_{ef}, of irreducible coherent pairs, such that $\{I_\alpha\}_\alpha$ and $\{J_\alpha\}_\alpha$ are partitions of $\overline{n}$. Denote $U_\alpha := \widehat{e}_{I_\alpha} = \widehat{f}_{J_\alpha}$. One has $\mathbb{C}^n = \oplus_\alpha U_\alpha$. This will be the decomposition associated to the edge ef. The decompositions associated to other edges are constructed in a similar way, and one arrives at the graph of the form:



(8)

with

$$\mathbb{C}^n = \oplus_\alpha U_\alpha = \oplus_\beta V_\beta = \oplus_\gamma W_\gamma = \oplus_\xi X_\xi = \oplus_\eta Y_\eta = \oplus_\zeta Z_\zeta.$$

The index set corresponding to the decomposition of $\mathbb{C}^n$ associated to the edge uv will be denoted as A_{uv}. Note now, that if $i \in I_\alpha$, $j \in J_{\alpha'}$ and $\alpha \neq \alpha'$, then $e_i \perp f_j$. Indeed, $f_j \in \widehat{f}_{J_{\alpha'}} = \widehat{e}_{I_{\alpha'}} \perp \widehat{e}_{I_\alpha} \ni e_i$. It means that for such i and j there exists no point of Δ of the form $(i, j, \cdot, \cdot)$. Similar statements can be formulated, of course, for other pairs of bases. Denote $\widetilde{e}_{\alpha,\gamma,\xi} := U_\alpha \cap W_\gamma \cap X_\xi$, $\widetilde{f}_{\alpha,\beta,\eta} := U_\alpha \cap V_\beta \cap Y_\eta$, $\widetilde{g}_{\beta,\gamma,\zeta} := V_\beta \cap W_\gamma \cap Z_\zeta$ and $\widetilde{h}_{\xi,\eta,\zeta} := X_\xi \cap Y_\eta \cap Z_\zeta$. Consider a set

$$\widetilde{\Delta} := \{(\alpha, \beta, \gamma, \xi, \eta, \zeta) | \widetilde{e}_{\alpha,\gamma,\xi}, \widetilde{f}_{\alpha,\beta,\eta}, \widetilde{g}_{\beta,\gamma,\zeta}, \widetilde{h}_{\xi,\eta,\zeta} \neq \{0\}\}.$$

It follows, that there is a natural map $\Delta \to \widetilde{\Delta}$. In particular, $\Delta \neq \emptyset$ implies $\widetilde{\Delta} \neq \emptyset$.

Let us return for a moment to the example when $n = 8$ and the bases $\{e_i\}_{i=1}^n$, $\{f_j\}_{j=1}^n$, $\{g_k\}_{k=1}^n$, $\{h_l\}_{l=1}^n$ are of the form $\{\Psi^\nu_{\sigma(\cdot)}\}_{\mu\in F_\nu}$, $\nu = e, f, g, h$. In this case, an orthogonal decomposition associated to every edge contains two terms, and the triple intersections of subspaces, associated to every vertex are one-dimensional. It means, that the property of $F_{\pi(\cdot)}$ can be reformulated as follows: for every $(\alpha, \beta, \gamma, \xi, \eta, \zeta) \in \widetilde{\Delta}$ the orthogonal projection of $F_{\pi(\cdot)}$ onto at least one of subspaces $\widetilde{e}_{\alpha,\gamma,\xi}$, $\widetilde{f}_{\alpha,\beta,\eta}$, $\widetilde{g}_{\beta,\gamma,\zeta}$ or $\widetilde{h}_{\xi,\eta,\zeta}$ is zero.

Now look at the general case. Imagine, that one would like to show that for any $\Phi \neq 0$ there exists $(i, j, k, l) \in \Delta$, such that Φ is not orthogonal to all the four vectors e_i, f_j, g_k, and h_l. In particular, there should exist $(\alpha, \beta, \gamma, \xi, \eta, \zeta) \in \widetilde{\Delta}$, such that Φ has a nonzero orthogonal projection onto all the four subspaces $\widetilde{e}_{\alpha,\gamma,\xi}$, $\widetilde{f}_{\alpha,\beta,\eta}$, $\widetilde{g}_{\beta,\gamma,\zeta}$, $\widetilde{h}_{\xi,\eta,\zeta}$. It is possible to give a general reason, why such a situation should always take place in the classical case, and why it may not take place in the quantum case: this is *"noncommutativity."* Observe the following. Let $\widehat{\pi}_L$ denote the orthogonal projector onto a linear subspace $L \subset \mathbb{C}^n$. Thus one has the orthogonal projectors $\widehat{\pi}_{U_\alpha}$, $\widehat{\pi}_{V_\beta}$, $\widehat{\pi}_{W_\gamma}$, $\widehat{\pi}_{X_\xi}$, $\widehat{\pi}_{Y_\eta}$, $\widehat{\pi}_{Z_\zeta}$. Note, that the projectors associated to one and the same vertex (rf. (8)), commute. Indeed, for example, $[\widehat{\pi}_{U_\alpha}, \widehat{\pi}_{V_\beta}] = 0$, since both U_α and V_β can be expressed as some linear spans over some e_i and $e_i \perp e_{i'}$, $i \neq i'$. On the other hand, if the projectors correspond to the edges, which have no common vertices, there is no general reason for them to commute. For example, in general, $[\widehat{\pi}_{U_\alpha}, \widehat{\pi}_{Z_\zeta}] \neq 0$.

Imagine now, that all the mentioned projectors do commute. Take any $\Phi \neq 0$ and consider the decomposition:

$$\Phi = \sum_{\alpha,\beta,\gamma,\xi,\eta,\zeta} \widehat{\pi}_{U_\alpha}\widehat{\pi}_{V_\beta}\widehat{\pi}_{W_\gamma}\widehat{\pi}_{X_\xi}\widehat{\pi}_{Y_\eta}\widehat{\pi}_{Z_\zeta}\Phi.$$

There exists at least one $(\alpha, \beta, \gamma, \xi, \eta, \zeta)$, such that the corresponding term in this sum is not zero. Since, due to the commutativity of the projectors, this term belongs to the intersection $\widetilde{e}_{\alpha,\gamma,\xi} \cap \widetilde{f}_{\alpha,\beta,\eta} \cap \widetilde{g}_{\beta,\gamma,\zeta} \cap \widetilde{h}_{\xi,\eta,\zeta}$, one has, in particular, $(\alpha, \beta, \gamma, \xi, \eta, \zeta) \in \widetilde{\Delta}$. Moreover, using the commutativity of the projectors, one deduces, that the orthogonal projection of Φ on each of the subspaces $\widetilde{e}_{\alpha,\gamma,\xi}$, $\widetilde{f}_{\alpha,\beta,\eta}$, $\widetilde{g}_{\beta,\gamma,\zeta}$ or $\widetilde{h}_{\xi,\eta,\zeta}$, is not zero. Indeed, for example, for $\widetilde{e}_{\alpha,\gamma,\xi}$ one has

$$\widehat{\pi}_{U_\alpha}\widehat{\pi}_{V_\beta}\widehat{\pi}_{W_\gamma}\widehat{\pi}_{X_\xi}\widehat{\pi}_{Y_\eta}\widehat{\pi}_{Z_\zeta}\Phi = \widehat{\pi}_{V_\beta}\widehat{\pi}_{Y_\eta}\widehat{\pi}_{Z_\zeta}[\widehat{\pi}_{U_\alpha\cap W_\gamma\cap X_\xi}\Phi] \neq 0,$$

and thus $\widehat{\pi}_{U_\alpha\cap W_\gamma\cap X_\xi}\Phi \neq 0$ (recall that $\widetilde{e}_{\alpha,\gamma,\xi} = U_\alpha \cap W_\gamma \cap X_\xi$).

This observation leads to the idea of how to treat the passage to the semiclassical limit. Let us imagine, that the subspaces $U_\alpha, \dots, Z_\zeta$ start to depend upon some formal small parameter $\varepsilon \to 0$: $U_\alpha = U_\alpha(\varepsilon), \dots, Z_\zeta = Z_\zeta(\varepsilon)$. The dimension n of $\mathbb{C}^n$ can also change with ε, $n = n(\varepsilon)$. We shall omit the argument ε in what follows. Assume, that the commutators between the projectors can be nonzero, but they are small as $\varepsilon \to 0$:

$$[\widehat{\pi}_{U_\alpha}, \widehat{\pi}_{Z_\zeta}] = O(\varepsilon), \quad [\widehat{\pi}_{V_\beta}, \widehat{\pi}_{Y_\eta}] = O(\varepsilon), \quad [\widehat{\pi}_{W_\gamma}, \widehat{\pi}_{X_\xi}] = O(\varepsilon). \tag{9}$$

(The estimate $O(\varepsilon)$ is to be understood in the sense of the operator norm). The rest of the commutators are equal to zero precisely. Assume also, that the number of terms in the decompositions $\mathbb{C}^n = \oplus_\alpha U_\alpha = \cdots = \oplus_\zeta Z_\zeta$, remains to be of order $O(1)$ as $\varepsilon \to 0$ (i.e., in general, A_{uv} also changes with ε, $A_{uv} = A_{uv}(\varepsilon)$, and $\#A_{uv}(\varepsilon) = O(1)$). Take now any $\Phi_\varepsilon \in \mathbb{C}^n$ such, that $\|\Phi_\varepsilon\| = O(1)$, $\varepsilon \to 0$. Consider the decomposition:

$$\Phi_\varepsilon = \sum_{\alpha,\beta,\gamma,\xi,\eta,\zeta} \widehat{\pi}_{U_\alpha}\widehat{\pi}_{V_\beta}\widehat{\pi}_{W_\gamma}\widehat{\pi}_{X_\xi}\widehat{\pi}_{Y_\eta}\widehat{\pi}_{Z_\zeta}\Phi_\varepsilon.$$

Since the left-hand side of this equation is of order $O(1)$ and in the right-hand side the number of terms is of order $O(1)$ as $\varepsilon \to 0$, for every ε one can choose $(\overline{\alpha}(\varepsilon), \ldots, \overline{\zeta}(\varepsilon))$ in such a way, that

$$\widehat{\pi}_{U_{\overline{\alpha}(\varepsilon)}}\widehat{\pi}_{V_{\overline{\beta}(\varepsilon)}}\widehat{\pi}_{W_{\overline{\gamma}(\varepsilon)}}\widehat{\pi}_{X_{\overline{\xi}(\varepsilon)}}\widehat{\pi}_{Y_{\overline{\eta}(\varepsilon)}}\widehat{\pi}_{Z_{\overline{\zeta}(\varepsilon)}}\Phi_\varepsilon = O(1).$$

From this one derives, that the projection of Φ_ε onto the triple intersections of subspaces of the form $U_{\overline{\alpha}(\varepsilon)} \cap V_{\overline{\beta}(\varepsilon)} \cap X_{\overline{\xi}(\varepsilon)}, \ldots$, are nonzero. Indeed, let us show this for the case of $U_{\overline{\alpha}(\varepsilon)} \cap V_{\overline{\beta}(\varepsilon)} \cap X_{\overline{\xi}(\varepsilon)}$. Using the commutation relations between the projectors, one obtains:

$$\begin{aligned} &\widehat{\pi}_{U_{\overline{\alpha}(\varepsilon)}}\widehat{\pi}_{V_{\overline{\beta}(\varepsilon)}}\widehat{\pi}_{W_{\overline{\gamma}(\varepsilon)}}\widehat{\pi}_{X_{\overline{\xi}(\varepsilon)}}\widehat{\pi}_{Y_{\overline{\eta}(\varepsilon)}}\widehat{\pi}_{Z_{\overline{\zeta}(\varepsilon)}}\Phi_\varepsilon \\ &\quad = \widehat{\pi}_{W_{\overline{\gamma}(\varepsilon)}}\widehat{\pi}_{Y_{\overline{\eta}(\varepsilon)}}\widehat{\pi}_{Z_{\overline{\zeta}(\varepsilon)}}[\widehat{\pi}_{U_{\overline{\alpha}(\varepsilon)} \cap V_{\overline{\beta}(\varepsilon)} \cap X_{\overline{\xi}(\varepsilon)}}\Phi_\varepsilon] + O(\varepsilon). \end{aligned}$$

(Note, that the number of times, that is necessary to apply the formulas (9), is equal to $O(1)$.) Since the left-hand side of this equation is of order $O(1)$, the expression in the square brackets in the right-hand side should also be of order $O(1)$. Note, that in particular this means, that $U_{\overline{\alpha}(\varepsilon)} \cap V_{\overline{\beta}(\varepsilon)} \cap X_{\overline{\xi}(\varepsilon)}$ is not trivial as $\varepsilon \to 0$. Similarly, one proves, that the projections of Φ_ε on other triple intersections, namely on $U_{\overline{\alpha}(\varepsilon)} \cap W_{\overline{\gamma}(\varepsilon)} \cap Y_{\overline{\eta}(\varepsilon)}$, $V_{\overline{\beta}(\varepsilon)} \cap W_{\overline{\gamma}(\varepsilon)} \cap Z_{\overline{\zeta}(\varepsilon)}$ and $X_{\overline{\alpha}(\varepsilon)} \cap Y_{\overline{\eta}(\varepsilon)} \cap Z_{\overline{\zeta}(\varepsilon)}$, are of order $O(1)$. In particular, this means that all this triple intersections are not trivial and thus $(\overline{\alpha}(\varepsilon), \ldots, \overline{\zeta}(\varepsilon))$ falls into the corresponding set $\widetilde{\Delta} = \widetilde{\Delta}(\varepsilon)$.

As one approaches the classical situation, there should appear a solution of the first part of the hidden variables problem. It means, that there should appear a space on which one can try to define a probability measure to construct a model of the experiment (this is the second part of the hidden variables problem). In general, such a probability measure need not exist. This is similar to the following fact. The Weyl symbol of a density matrix for a quantum system is a real distribution on a phase space of the associated classical system. It may be viewed as an analog of a phase-space density distribution in classical statistical mechanics. In general it is not positively defined, but one can prove (under some assumptions), that in the classical limit the Lebesgue measure of the set of points where it is negative, disappears. It means, that in the classical limit one finally has a measurable space and a probability measure.

Note, that one may naturally imitate the passage to the semiclassical limit with an abstract family of objects $\{X_\varepsilon\}_\varepsilon$ of $\mathcal{C}_1$. In particular, to construct an analog of $\widetilde{\Delta}(\varepsilon)$, one has to consider for every given $M, N \in \mathcal{M}_{X_\varepsilon}$, $M \neq N$, a decomposition of $\mathbf{1}_{X_\varepsilon}$ into an $\oplus$-sum with terms given by the elements of $\mathrm{Min}((M \cap N)\setminus\{\mathbf{0}_{X_\varepsilon}\}; \preccurlyeq)$, where $\preccurlyeq$ is the standard partial order. Since the notion of the passage to such a limit refers to the two problems, namely, the existence of a measurable space and the existence of a probability measure, quantization should also be viewed as a compound notion: there is a problem of 'twisting' the multiplication of the classical observables (the twisted product should be consistent with the equation for an analog of the classical probability distribution), and a problem of "distorting" the underlying measurable space itself.

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